Topological Hochschild homology, traces and higher categories

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Abstract

This is the typed notes of a lecture given in Bielefeld starting from Winter 2025/2026, as a follow-up to the course Higher categories and algebraic K-theory III taught by Fabian Hebestreit. Their goal is to explain a modern point of view on THH, and more generally $\mathcal{V}$-linear Hochschild homology through higher (enriched) category theory.

In a second time, we build a bridge between the material of this course and the previous one, by investigating the relationship between algebraic K-theory and topological Hochschild homology, a subject which is often called "trace methods".

Contents

The main character of these notes is THH, topological Hochschild homology, a spectrum which can be associated to a ring, a ring spectrum, a stable category with even suitable coefficients. THH is a very rich object: it enjoys an interesting functoriality, has plenty of mysterious extra structure, is linked with many other invariants of interests and comes with a famed history. As our point of view will be quite modern, we want to recall a few elements of history at the very beginning.

Before topological Hochschild homology, there was simply Hochschild homology, which we will denote $\mathrm{HH}_{\mathbb{Z}}$. Given a ring R and a R-bimodule M, $\mathrm{HH}_{\mathbb{Z}}(R, M)$ is traditionally as the Tor-groups of R and M viewed as modules over $R \otimes R^{\mathrm{op}}$; our higher categorical point of view allows to simply write

$$\mathrm{HH}_{\mathbb{Z}}(R, M) := R \otimes^{\mathrm{L}}_{\mathbb{Z}} M$$

where for the first and last time, we added a superscript L to insist on the fact that this tensor is derived, that is, taken in the presentable stable category $D(\mathbb{Z})$ of derived $\mathbb{Z}$-modules which we will simply denote $\mathrm{Mod}(\mathbb{Z})$, keeping in with modern fashion.

The name comes from Hochschild, who introduced it in a paper as the homology of an explicit complex (the so-called Hochschild complex). In modern terms, it is obtained via the Dold-Kan correspondence from the following simplicial object:

$$M \otimes R^{\otimes n} \rightrightarrows \dots \rightrightarrows M \otimes R \rightrightarrows M$$

using the R-linear multiplication map on M on the k^{th}-component. Note in particular that $\pi_0 \mathrm{HH}_{\mathbb{Z}}(R, M) \simeq R/[R, M]$, and the canonical map $\mathrm{tr} : \mathrm{Proj}(R)^{\simeq} \to R/[R, R]$ factors through $K_0(R)$, because it is in particular additive. In fact, the trace map lifts to the whole spectrum, to something called the *Dennis trace map*:

$$K(R) \longrightarrow \mathrm{HH}_{\mathbb{Z}}(R, R)$$

Even for non-K-theorists, Hochschild homology is quite an interesting object. For a smooth commutative algebra A over a field k of characteristic zero, $\pi_n \operatorname{HH}_{\mathbb{Z}}(A/k, A/k)$ coincides with the Kähler differentials $\Omega^n_{A/k}$, and for general non-smooth A, receives at least a comparison map.

It also comes with an action of the circle S^1, related to the de Rham differentials $\Omega^n_{A/k} \to \Omega^{n+1}_{A/k}$ (we refer to Matthew Mororw's notes for more details in this direction). This S^1-action only exist when the bimodule is the ring itself, and can be understood from the perspective of cyclic objects, as the geometric realization of cyclic object carries canonically such an action.

Taking homotopy fixed points and the Tate cohomology (in the $\mathbb{Z}$-linear world) for the S^1-action yields spectra called *(negative) cyclic homology* HC^- and HP. There is a map $\mathrm{HC}^- \to \mathrm{HP}$ whose fiber we call HC, the *cyclic homology* — traditionally, we should incorporate a shift because the norm map for S^1 has a shift (see Corollary I.4.3 of [NS17]):

$$\Sigma(-)_{\mathrm{h}S^1} \xrightarrow{\mathrm{Nm}_{S^1}} (-)^{\mathrm{h}S^1} \longrightarrow (-)^{\mathrm{t}S^1}$$

so that HC coincides with homotopy orbits for the S^1-action but this introduce an annoying shift in the notation everywhere else. These invariants were discovered first by Connes and Tsygan, without quite realizing the S^1-action at first, and it is under Connes' impulse that the cyclic category was introduced to formalize this action and reinterpret the earlier construction in this framework.

The trace map for $\mathrm{HH}_{\mathbb{Z}}$ is not very interesting because Hochschild homology is often far too simple to tell interesting things in K-theory: for instance there is no higher Hochschild homology for $\mathbb{Z}$ so this map loses the information on higher K-groups of $\mathbb{Z}$, and similarly for $\mathbb{F}_p$, $\mathrm{K}(F_p)$ is in odd degrees and $\mathrm{HH}_{\mathbb{Z}}(\mathbb{F}_p)$ is even. However, this trace map is S^1-equivariant for the trivial action on K-theory. In particular, it lifts to a map $\mathrm{K} \to \mathrm{HC}^-$ and rationally, it also vanishes on HP hence lifts to $\mathrm{K} \otimes \mathbb{Q} \to \mathrm{HC} \otimes \mathbb{Q}$. This refined *cyclic trace map* is actually able to capture more on K-theory. A result of Goodwillie [Goo86] states that if $R \to S$ is surjective with nilpotent kernel, then

$$\mathrm{fib}(\mathrm{K}(R) \to \mathrm{K}(S)) \longrightarrow \mathrm{fib}(\mathrm{HC}(R) \to \mathrm{HC}(S))$$

is a rational equivalence (i.e. an equivalence after tensoring with $\mathbb{Q}$). This is quite a striking result, as computations in K-theory are really hard, whereas HC is a manageable object to compute. Unfortunately, it just breaks down away from characteristic zero. This is where THH enters the story.

The insight, due to Goodwillie and Waldhausen, is that K-theory, unlike HH, is not really a "linear" object, e.g. $\mathrm{K}(\mathbb{F}_p)$ is not a $\mathbb{F}_p$-module. It mostly lives over the initial (non-zero) ring ... but this is *not(!)* $\mathbb{Z}$ in homotopy theory, but the sphere spectrum $\mathbb{S}$. They wondered if there was a "topological" refinement of HH (in the sense that it understood more than just $\pi_0\mathbb{S} \simeq \mathbb{Z}$ but the topology above) and this replacement should make the statements hold integrally.

In fact, it was known that *stable K-theory*, the invariant obtained from K-theory by forcefully adding a dependence in the bimodule variable via the square-zero extension $\mathrm{K}(R \oplus M)$ and then forcing it to be M-linear, was rationally Hochschild homology and it was expected that the integral object was this topological Hochschild homology, a conjecture that made it into Goodwillie's 1990 ICM address.

Of course, this predates more higher categorical technology so it took Bökstedt some amount of effort to define properly THH, and study the extra structure — with Hsiang and Madsen in [BHM93], they described that not only did $\mathrm{THH}(R)$ have a S^1-action, it also had a cyclotomic structure which in modern terms we would describe as S^1-equivariant maps

$$\phi_p : \mathrm{THH}(R) \longrightarrow \mathrm{THH}(R)^{\mathrm{tC}_p}$$

Using this structure, one can form topological version of the periodic and negative theories we introduced earlier, namely we let $\mathrm{TC}^-(R) := \mathrm{THH}(R)^{\mathrm{h}S^1}$, $\mathrm{TP}(R) := \mathrm{THH}(R)^{\mathrm{t}S^1}$, but the correct replacement of HC actually involves the cyclotomic structure. In formula, following the Nikolaus–Scholze approach of [NS17], one lets:

$$\mathrm{TC}(R) := \mathrm{Eq}\left(\mathrm{THH}(R)^{\mathrm{h}S^1} \underset{(\phi_p^{\mathrm{h}S^1})}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_{p \text{ prime}} (\mathrm{THH}(R)^{\mathrm{tC}_p})^{\mathrm{h}S^1} \right)$$

The resulting invariant is called *topological cyclic homology*. About at the same time, Dundas–McCarthy proved in [DM94] that THH was indeed stable K-theory for connective rings and connective bimodules, and after some more efforts, they produced an integral version of Goodwillie's theorem, namely that if $R \to S$ is map of connective ring spectra such that on π_0, it is surjective with nilpotent kernel, then

$$\mathrm{fib}(\mathrm{K}(R) \to \mathrm{K}(S)) \xrightarrow{\simeq} \mathrm{fib}(\mathrm{TC}(R) \to \mathrm{TC}(S))$$

is an equivalence. The proof of this result is quite technical, and relies on both various simplicial comparisons and the calculus of functors of Goodwillie — something that Goodwillie had already envisioned for his result in [Goo86] even if he did not use it in the end. This result is particularly key to compute K-theory of more complicated rings, like $\mathbb{Z}/p^n\mathbb{Z}$ when $n \geq 2$, see [AKN24].

The goal of this course, or what we want to achieve at the end of multiple courses following one another, is to *explain* this result, and the word has been italicized because we do not simply want to give a presentation of the proof with minor modern improvement but truly a different treatment of it, which follows the ideas of the series of papers [HNS24, HNRS26a, HNRS26b] — which are also currently not all been publicly released.

There are two major differences we want to implement: the first is to move away from rings, or even ring spectra and try to understand this story at the level of stable categories, sometimes idempotent-complete or even large dualizable following insights of [Efi24]. In K-theory, this has always been somewhat standard ever since Quillen's seminal work on higher algebraic K-theory [Qui73] but references for THH of stable categories are few and far between. We claim that done correctly, this will allow, just as in K-theory, to turn THH from an object realized by a certain construction and the structure therein inherited from special properties of this construction, into an object having a universal property and us being able to prove central features of THH via the study of the often simpler property.

In THH, unlike in K-theory, it is central to implement this *with coefficients*. These coefficients, which generalize bimodules over a ring, are bi-exact functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$. We will also explain why these are naturally the "coefficients of a linear theory" over $\mathbf{Cat}^{\mathrm{Ex}}$ — by this we mean functors $F(\mathcal{C}, -)$ where the blank variable is colimit-preserving or at least exact — by identifying them as $\mathcal{C}$ varies with the category $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, the *tangent bundle of* $\mathbf{Cat}^{\mathrm{Ex}}$ which is the abstract category of coefficients of linear theories over $\mathbf{Cat}^{\mathrm{Ex}}$.

In this world, we will furnish a universal property for THH, which will use that $\mathrm{THH}(\mathcal{C}, M)$ is linear in the M-variable and the other key feature of THH we have not mentioned: its invariance under cyclic permutations. More precisely if M is a $(\mathcal{C}, \mathcal{D})$-bimodule and N a $(\mathcal{D}, \mathcal{C})$-bimodule, then there is an equivalence:

$$\mathrm{THH}(\mathcal{C}, M \otimes_{\mathcal{D}} N) \simeq \mathrm{THH}(\mathcal{C}, N \otimes_{\mathcal{D}} M)$$

This cyclic invariance is one of the defining feature of the trace. The reader fond of linear algebra might know for instance that a linear form $f : M_n(\mathbb{R}) \to \mathbb{R}$ which has the cyclic invariance is necessarily a multiple of the trace, namely $f(-) = f(E_1)\,\mathrm{tr}(-)$.

The notion of trace is one that can be defined extremely generally. We will show that THH *is* a trace, in the category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ of (large) presentable stable categories, which will take us into a expository panorama of large categories in the higher world. In fact, following Ramzi in his thesis, we will show that the uniqueness characterization of the trace *lifts* to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, namely the functor

$$\mathrm{ev}_{\mathbb{1}} : \mathrm{Fun}^{\mathrm{cyc},\mathrm{fbw-L}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \longrightarrow \mathcal{E}$$

which evaluates at the unit of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ a cyclic-invariant, colimit-preserving in the coefficients, functor to a presentable stable $\mathcal{E}$, is an equivalence with inverse $X \mapsto X \otimes \mathrm{THH}$.

In fact, there is a refinement of this story that is central to trace methods in K-theory. Let us first recall some linear algebra: given a real matrix M over $\mathbb{R}$, one can compute the whole Taylor tower of $\det(I + tM)$. In fact, it is even easier to express after passage to the logarithm:

$$\ln\det(I + tM) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n}\,\mathrm{tr}(M^n)t^n$$

We claimed that THH was a refinement of the trace and our refinement of $\ln\det(I+tM)$ is the fiber $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, M) := \mathrm{fib}(\mathrm{K}(\mathcal{C}\oplus M) \to \mathrm{K}(\mathcal{C}))$ where $\mathcal{C}\oplus M$ is a categorical version of the square-zero extension, details of which we won't go into now. Note that as the name suggest, $\mathrm{K}^{\mathrm{cyc}}$ has the cyclic invariance of the trace (in fact, this is also true of $\det(I+M)$ and is known under the name of *Weinstein-Aronszajn identity*). In particular, forcefully imposing cyclic K-theory to commute with colimits in the M variable will give a point in $\mathrm{Fun}^{\mathrm{cyc,fbw-L}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$, i.e. this derivative is of the form $X \otimes \mathrm{THH}$ and $X \simeq \mathbb{S}$ by a previously mentioned result of Dundas-McCarthy.

More generally, one can show that a n-excisive, finitary, additive, cyclic invariant $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ promotes to n-truncated polygonic objects in $\mathcal{E}$, i.e. that one can record functorially the data $F(\mathcal{C}, M^{\otimes k})$ for $1 \leq k \leq n$ and they are related by maps

$$\phi_{k,l} : F(\mathcal{C}, M^{\otimes k}) \longrightarrow F(\mathcal{C}, M^{\otimes kl})^{\tau \mathrm{C}_l}$$

which are C_k-invariant and exist for $kl \leq n$. The target is the *proper* Tate construction, i.e. the Tate construction with respect to the family of proper subgroups of C_l instead of the usual (which is with respect to the trivial family of subgroups). This provides a functor

$$\mathrm{Fun}^{\mathrm{cyc,fbw-}n\mathrm{exc,add,}\omega}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \longrightarrow \mathrm{Pgc}_{\leq n}(\mathcal{E})$$

which evaluates at the unit of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ the refined functor valued in polygonic objects. The extra hypotheses (finitary, additive) are precisely added so that this functor is still an equivalence. The inverse is given in formula by $X \mapsto \mathrm{TR}_n(X \otimes \mathrm{THH}(-))$ where the tensor product is using that $\mathrm{Pgc}_{\leq n}(\mathcal{E})$ is tensored over $\mathrm{Pgc}_{\leq n}(\mathrm{Sp})$, that THH admits such a structure and $\mathrm{TR}_n : \mathrm{Pgc}_{\leq n}(\mathcal{E}) \to \mathcal{E}$ is the right adjoint of the trivial functor.

In particular, TR_n is trying to glue back the extra data supplied by the polygonic spectra, in a way not too dissimilar to truncating the sum of traces in the Taylor tower of $\ln\det(I+tM)$. In fact, taking the n-excisive approximation of cyclic K-theory gives a functor $\mathbf{TCat}^{\mathrm{Ex}} \to \mathrm{Sp}$ with all of the extra properties which coincides with $\mathrm{TR}_n(\mathrm{THH})$, i.e. up to some extension problems which mean we cannot write a direct sum, the formula for $\ln\det(I+tM)$ holds also in the world of stable categories with coefficients.

In fact, and at least for the case of square-zero extensions, the Dundas-Goodwillie-McCarthy theorem can be understood as a phenomenon of both cyclic K-theory and cyclic TC converging to the limit of their Taylor tower, which also happen to coincide.

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1 How to tame your large category

Large categories can be scary. The goal of this section is to give the reader some tools to turn their large, smelly, hirsute category into a well-behaved, groomed and all-together *presentable* category.

1.1 The amazing category ~~of spaces of homotopy-types of anima of groupoids~~ $\mathcal{S}$

Let us begin by a example, the poster child of a nice, large category: the category $\mathcal{S}$ of what we will call spaces or groupoids but feel free to use any other word you prefer, like anima or homotopy-type. The category $\mathcal{S}$ is the full subcategory of **Cat** spanned by those categories where every arrow is invertible.

Proposition 1.1.1 The inclusion $\mathcal{S} \to \mathbf{Cat}$ admits both a left and a right adjoint. The former is denoted $|\cdot|$, and computed by forming the localization at all arrows and the latter, denoted $(-)^{\simeq}$, is obtained as the wide subcategory spanned by invertible arrows.

In particular, the above provides a rather robust way of computing colimits and limits in $\mathcal{S}$, since one can use the machinery developed for categories. We recall the following statement, which is paramount to compute colimits of categories and was proven in Fabian's earlier lecture.

Lemma 1.1.2 Let $F : \mathcal{C} \to \mathbf{Cat}$ and denote $\mathrm{Un}(F) \to \mathcal{C}$ the cocartesian unstraightening of F. Then, $\operatorname{colim} F$ is the localisation of $\mathrm{Un}(F)$ at the cocartesian edges.

As a sanity check, remark that if F is space-valued, then every arrow in $\mathrm{Un}(F)$ factors as a cocartesian edge followed by an equivalence, so $\operatorname{colim} F$ is indeed a space.

To compute limits in **Cat**, one must take the category of sections of the unstraightening of a functor instead, and then take the full subcategory spanned by those γ such that for every $\alpha : i \to j$, the induced map $F(\alpha)(\gamma(i)) \to \gamma(j)$ is an equivalence (or the other way around, depending of whether one takes the cocartesian or the cartesian unstraightening). The reader who is not familiar with these ideas is invited to try to compute for instance pullbacks of categories this way, as we will often use the concreteness of this construction.

Corollary 1.1.3 The category $\mathcal{S}$ is complete and cocomplete.

Let us further analyse colimits in $\mathcal{S}$. For this, we will need an intermediary result; if $\mathcal{C}$ is a category, we write $\mathcal{P}(\mathcal{C})$ for the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves and $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ for the Yoneda embedding.

Lemma 1.1.4 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with target a cocomplete category, and let $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the Yoneda embedding. Then, the functor $j_!F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ preserves colimits.

Proof. We follow roughly Theorem 8.4.3.5 in [Lur18a, Tag 03WH]. Since we can test along mapping spaces (i.e. it suffices to show that $\mathrm{Map}(j_!F(-), X)$ sends colimits to limits), we reduce without loss of generality to the case where $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$.

Note that $(j_!)^{\mathrm{op}}$ identifies with the functor which right Kan extend along j^{op}, which is just the Yoneda embedding of $\mathcal{C}^{\mathrm{op}}$. Under this identification, we have to justify that this functor restricts to

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \longrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}), \mathcal{S}^{\mathrm{op}})$$

i.e. that if $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a presheaf, then j_*F sends colimits to limits. But by the Yoneda lemma, there is an equivalence

$$j_*F \simeq \mathrm{Nat}(-, F)$$

since both sides have the same universal property.

A different proof of this claim is to remark that $j_!F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ has a right adjoint given by

$$R_F : X \in \mathcal{D} \longmapsto \mathrm{Map}_{\mathcal{D}}(F(-), X) \in \mathcal{P}(\mathcal{C})$$

which can be checked by the local criterion for adjunctions. □

Proposition 1.1.5 Every object in $\mathcal{S}$ is a colimit of $*$. In fact, $\mathcal{S}$ is *freely* generated under colimits, in the sense that the inclusion $i : \{*\} \to \mathcal{S}$ induces an equivalence

$$i^* : \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$$

for every cocomplete $\mathcal{C}$, where $\mathrm{Fun}^{\mathrm{L}}$ denotes the full subcategory of colimit-preserving functors.

Proof. Using Lemma 1.1.2, the first claim is immediate as any $X \in \mathcal{S}$ is also the unstraightening of the associated functor $\mathrm{cst}(*) : X \to \mathcal{S}$ of the constant functor equal to $*$.

Let us also include a more "topological" explanation. Recall that up to weak equivalence, every (nice-enough) topological space is a CW-complex, with possibly infinitely many cells in each dimension. In particular, each cell is built out of spheres $S^n := \Sigma^n S^0$ where $S^0 := \{*\} \coprod \{*\}$ and disks which are contractible i.e. homotopic to a point. Since the gluing in CW-complexes happens along cofibrations, any presentation of a CW-complex gives rises to a colimit-presentation of the associated homotopy type.

We now prove that i is an equivalence. First recall from say [Lur08, Proposition 4.3.3.7] that

$$i^* : \mathrm{Fun}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$$

has a left adjoint $i_!$ which is fully-faithful and given by the left Kan extension functor. Unravelling the formula for left Kan extension, for a point $A : * \to \mathcal{C}$, $i_! A$ is the functor described pointwise by

$$X \mapsto \operatorname*{colim}_{p:*\to X} A(*)$$

which we will often write $X \otimes A(*)$. This colimit does indeed exist since $\mathcal{C}$ is cocomplete and $i_!$ is fully-faithful because i is (see §4.3.2 of [Lur08]). Because colimits commute with other colimit or more generally, thanks to the lemma below, this adjoint lands in the full subcategory $\mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{C})$ and therefore the whole adjunction descends.

In particular, as the adjoint of a fully-faithful functor, i^* is a localisation, namely at the collection of arrow $\mathcal{W} := \{i_! i^*(F) \longrightarrow F\}$ — by this, we mean that any functor Φ which inverts those arrows must factor essentially uniquely through i^*: this is obvious since the collection of arrows gives the factorization $(\Phi \circ i_!) \circ i^*$. Now notice that i^* is conservative: if $F \to G$ is a natural transformation of colimit-preserving functors $\mathcal{S} \to \mathcal{C}$ such that $F(*) \to G(*)$ is an equivalence, we claim that $F(X) \to G(X)$ is also always an equivalence. This follows easily from choosing a presentation of X as a colimit of $*$ which exists by the first part.

To conclude, we remark that a conservative localisation is necessarily a localisation at no non-trivial arrows (or directly that a functor which is conservative and has a fully-faithful adjoint is an equivalence from the triangle identities), hence an equivalence. □

In fact, the proof of Proposition 1.1.5 generalizes to the following:

Proposition 1.1.6 Let $\mathcal{D}$ be a cocomplete category and $\mathcal{C}$ a small category, then, restriction along the Yoneda lemma induces an equivalence:

$$j^* : \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

We say that $\mathcal{P}(\mathcal{C})$ is *freely generated by* $\mathcal{C}$ *under colimits.*

Proof. Since $\mathcal{D}$ has small colimits, j^* has a left adjoint $j_!$ given by left Kan extension along j, which indeed lands in colimit-preserving functor and is fully-faithful since the Yoneda lemma guarantees that j is fully-faithful. Hence, as in Proposition 1.1.5, j^* is a localisation and it suffices to check that it is conservative and by similar arguments, this reduces to the fact that every $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a colimit of representable functors.

Indeed, there is a map of spaces, natural in X, which we can obtain by the evaluation of the counit of the above adjunction for the functor $\mathrm{id} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$,

$$\operatorname*{colim}_{j(Y)\to\phi} \mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \phi(X)$$

We claim this map is an equivalence. Note that this colimit is indexed by $\mathcal{P}(\mathcal{C})_{/\phi} \times_{\mathcal{P}(\mathcal{C})} \mathcal{C}$ which by the Yoneda lemma, corresponds to the cartesian unstraightening of the functor $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. But the functor $\mathrm{Map}(X, p(-)) : \mathrm{Un}^{cart}(\phi) \to \mathcal{S}$ factors through the projection $p : \mathrm{Un}^{cart}(\phi) \to \mathcal{C}$.

We can use Lemma 1.1.2 to compute this colimit. The cocartesian unstraightening of the composite $\mathrm{Map}(X, p(-))$ is given by $\mathcal{C}_{X/} \times_{\mathcal{C}} \mathrm{Un}^{cart}(\phi)$; this category receives a map from $\phi(X)$ thanks to the commutative diagram:

$$\begin{array}{ccc} \phi(X) & \xrightarrow{\mathrm{cst}(\mathrm{id}_X)} & \mathcal{C}_{X/} \\ \downarrow{\scriptstyle\subset} & & \downarrow \\ \mathrm{Un}^{cart}(\phi) & \longrightarrow & \mathcal{C} \end{array}$$

Now, $\phi(X)$ is a space so it suffices to argue that $\phi(X) \to \mathcal{C}_{X/} \times_{\mathcal{C}} \mathrm{Un}^{cart}(\phi)$ is a weak homotopy equivalence. This follows from the fact that this functor has an adjoint, which we can describe as sending $(f : X \to Y, y \in \phi(Y))$ to the $\phi(f)(y) \in \phi(X)$ — this defines a left adjoint because $(\mathrm{id}_X : X \to X, x \in \phi(X))$ is initial in each slice. This concludes. □

In particular in the proof, we obtained:

Corollary 1.1.7 Every presheaf is a colimit of representable presheaves, i.e. the image of j generates $\mathcal{P}(\mathcal{C})$ under colimits.

Another proof of the above follows from the formula of Lemma 1.4.22. We now turn to:

Definition 1.1.8 A *finite space* is an object $X \in \mathcal{S}$ which can be obtained as a finite colimit of $*$, i.e. in the smallest full subcategory of $\mathcal{S}$ closed under coproducts and pushouts and containing the point.

Note that it is hard to not be self-referential in defining the finiteness notion. The above is not but it was implemented the fact that iterated coproducts and pushouts produce all finite diagrams, which one might want as a property and not a definition.

An equivalent definition is that a category $\mathcal{C}$ is finite if and only if there exists a simplicial set weakly-equivalent to $\mathcal{C}$ with finitely many non-degenerate simplices. It holds that the category of finite categories is the smallest closed under pushouts and coproducts and containing both $*$ and $\{0 \to 1\}$. Note that since $\mathcal{S} \to \mathbf{Cat}$ preserves colimits, finite spaces are also legitimate finite categories and they span the further subcategory which is only generated by $*$; in particular, finite spaces are therefore equivalently those who can be modelled by Kan complexes with finitely many non-degenerate simplices and those obtained by finitely many pushouts and coproducts out of $*$.

We now want to explain what ones needs to do to recover the whole category $\mathcal{S}$ from its finite objects.

Definition 1.1.9 A diagram category I is called *filtered* if for every finite category $\mathcal{C}$, the diagonal functor $\mathrm{cst} : I \to \mathrm{Fun}(\mathcal{C}, I)$ sending i to the constant functor $\mathcal{C} \to I$ with value i is cofinal.

Differently stated thanks to Quillen's Theorem A [Lur08, Theorem 4.1.3.1], $\mathrm{cst} : I \to \mathrm{Fun}(\mathcal{C}, I)$ is cofinal if for every $f : \mathcal{C} \to I$, the category of diagrams $\mathrm{Fun}(\mathcal{C}, I)_{f/} \times_{\mathrm{Fun}(\mathcal{C},I)} I$, whose objects are natural transformations $\{f \longrightarrow \mathrm{cst}(i)\}$ and maps are induced by maps $i \to j$ in I making the associated diagrams commute, is weakly contractible.

■ **Example 1.1.10** Right adjoint functors are always cofinal by virtue of the categories required to be weakly contractible having an initial object, hence categories with finite colimits are filtered. ■

Remark 1.1.11 Actually, it suffices that each $\mathrm{Fun}(\mathcal{C}, I)_{f/} \times_{\mathrm{Fun}(\mathcal{C},I)} I$ is non-empty for it to be weakly contractible (see Proposition 9.1.1.18/Tag 02PJ of [Lur18a]).

In particular, since cofinal maps are weak equivalences, I is non-empty using the case $\mathcal{C} = \emptyset$. Moreover, using the case of finite sets, it follows that one can find a cone point for every pair of objects of I as well as an equalizing morphism for any two pair of morphisms. In particular, filtered 1-categories are filtered in the higher sense as well.

Writing a space as the filtered colimit of a finite skeleta, we get:

Corollary 1.1.12 Every object $X \in \mathcal{S}$ is a filtered colimit of finite spaces.

More is actually true, as we will soon show, but first let us introduce another notion, which we will quickly relate to finiteness:

Definition 1.1.13 A space X is *compact* if the functor $\mathrm{Map}(X, -) : \mathcal{S} \to \mathcal{S}$ commutes with filtered colimits.

■ **Example 1.1.14** The empty space $\emptyset$ is compact. The point $*$ is compact. ■

In fact, it is possible to recognize filtered categories by how the colimit functor valued in $\mathcal{S}$ behaves:

Proposition 1.1.15 In $\mathcal{S}$, filtered colimits commute with finite limits. In particular, finite spaces are compact.

Proof. Let us explain quickly the second part: as a functor in X, $\mathrm{Map}(X, -)$ sends finite colimits to finite limits. In particular, a finite colimit of compact objects stays compact (note that this actually holds for any category $\mathcal{C}$, as it only uses the commutation at the target). This concludes using the previous example.

For any finite diagram I, any filtered J and any functor $X : I \times J \to \mathcal{S}$, there is a natural map

$$\eta : \operatorname*{colim}_{j \in J} \lim_{i \in I} X(i,j) \longrightarrow \lim_{i \in I} \operatorname*{colim}_{j \in J} X(i,j)$$

and we have to check it is an equivalence. Note that we can reduce to the case where the finite limit is a pullback or terminal. The latter case is straightforward by virtue of filtered categories being contractible.

We will not give a proof for the case of pullbacks, but let us sketch a strategy. Because we have access to particularly explicit descriptions in the case of sets, it is easier to check that this property holds there. One strategy, which is the one of [Lur08], is therefore to push this fact through to nice topological spaces and then through the localisation, see for instance [Lur08, Proposition 5.3.3.3]. Another way of presenting this idea is through [Lur18a, Tag 05XW], which is less model-dependant. □

This makes $\mathcal{S}$ compactly-generated, i.e. every space is a filtered colimit of compact spaces; we will return to this property later. In a different direction, let us also say that this commutation property of filtered colimits is an equivalent characterization of filtered diagrams:

Corollary 1.1.16 A category J is filtered if and only if the functor $\operatorname{colim}_J : \operatorname{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves finite limits.

Proof. Given the above, we are reduced to prove that if colim_J preserves finite limits, then the functor $\operatorname{cst}_* : J \to \operatorname{Fun}(X, J)$ is cofinal for every finite category K. As we explained earlier, it suffices to check that for every $F : K \to J$, the category $\operatorname{Fun}(K, J)_{F/} \times_{\operatorname{Fun}(K,J)} J$ is weakly contractible. Another description of this category is the unstraightening of the functor $j \in \mathcal{J} \mapsto \operatorname{Nat}(F, \operatorname{cst}(i)) \simeq \lim_{k \in K^{\mathrm{op}}} \operatorname{Map}(F(k), j)$.

Now, since K^{op} is again finite, we know that

$$\eta : \operatorname*{colim}_{j \in J} \lim_{k \in K^{\mathrm{op}}} \operatorname{Map}(F(k), j) \longrightarrow \lim_{k \in K^{\mathrm{op}}} \operatorname*{colim}_{j \in J} \operatorname{Map}(F(k), j)$$

is an equivalence. In particular, the left hand side is also the localisation of $\operatorname{Fun}(K, J)_{F/} \times_{\operatorname{Fun}(K,J)} J$ at the cocartesian arrows by Lemma 1.1.2, which we precisely want to show is equivalent to a point.

To conclude, it suffices to remark that $\operatorname{colim}_{j \in J} \operatorname{Map}(x, j)$ is always contractible for any $x \in J$. Indeed, $\operatorname{Map}(x, -)$ is classified by the cocartesian fibration $J_{x/} \to J$ and $J_{x/}$ has an initial object, hence becomes contractible when inverting all its cocartesian edges. □

Let us also include a more pedestrian way of proving the last claim: note that there is a point in each $\operatorname{colim}_{j \in J} \operatorname{Map}(F(k), j)$ induced by $\mathrm{id}_{F(k)} : F(k) \to F(k)$. These lift to a point in the limit over K^{op} by functoriality and therefore we have a point $X \in \operatorname{colim}_{j \in J} \lim_{k \in K^{\mathrm{op}}} \operatorname{Map}(F(k), -)$. In consequence, since J is filtered and $*$ is compact, there is $j \in J$ such that $X \in \lim_{k \in K^{\mathrm{op}}} \operatorname{Map}(F(k), j)$ and using the projection maps of the limits, this endows j with the structure of a cone point to F, i.e. there is a map $F \to \operatorname{cst}(j)$. In particular, we have found our category to be non-empty. Being more careful we could show that it is connected, and so forth to get the result. This precisely what the unstraightening captures in a rigorous manner.

Corollary 1.1.17 An object $X \in \mathcal{S}$ is compact if and only if it is a retract of a finite space.

Proof. Compact objects are closed under retracts: indeed if X is compact and A is a retract of X, then there is a diagram:

$$\begin{array}{ccc}
\operatorname{colim}_{i \in I} \operatorname{Map}(A, Z_i) & \longrightarrow & \operatorname{Map}(A, \operatorname{colim}_{i \in I} Z_i) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i \in I} \operatorname{Map}(X, Z_i) & \xrightarrow{\simeq} & \operatorname{Map}(X, \operatorname{colim}_{i \in I} Z_i) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i \in I} \operatorname{Map}(A, Z_i) & \longrightarrow & \operatorname{Map}(A, \operatorname{colim}_{i \in I} Z_i)
\end{array}$$

which exhibits the colimit-comparison map of A as a retract of an equivalence, hence an equivalence again.

Now, given a compact space X, write $X \simeq \operatorname{colim}_{i \in I} X_i$ with I filtered and the X_i finite. Then, $\mathrm{id}_X : X \longrightarrow X \simeq \operatorname{colim}_{i \in I} X_i$ must factor through one of the X_i. This exhibits X as a retract of X_i which concludes. $\square$

Warning 1.1.18 Retracts of finite spaces need not be finite again. In general, retracts of finite spaces are called *finitely-dominated*. A Theorem of Wall, called Wall's finiteness obstruction, and related to Thomason's classification theorem shows that for a finitely-dominated space X, there is a class in $\widetilde{K_0}(\mathbb{S}[\Omega X])$ which vanishes if and only if X is finite.

Corollary 1.1.12 admits the following strengthening; in fact let us note that even if we present the result and the proof for $\mathcal{S}$, it holds *mutatis mutandis* when replacing $\mathcal{S}$ by $\mathcal{P}(\mathcal{C})$ for some small $\mathcal{C}$ and $\mathcal{S}^{\mathrm{fin}}$ by the full subcategory of $\mathcal{P}(\mathcal{C})$ containing the image of the Yoneda and stable under finite colimits.

Proposition 1.1.19 Suppose $\mathcal{C}$ has filtered-colimits, and write $i : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$ for the inclusion of the full subcategory spanned by finite spaces. Then,

$$i^* : \mathrm{Fun}^{\omega}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C})$$

is an equivalence, where the superscript ω denotes the full subcategory of *finitary*, i.e. filtered-colimit preserving functors. Its inverse is given by left Kan extension along i.

Proof. By Corollary 1.1.12, the above functor is conservative. We check that it has a left adjoint which is fully-faithful. In fact, we claim that this left adjoint is simply given by left Kan extension along i, which is automatically fully-faithful since i is. This follows from checking that if $f : \mathcal{S}^{\mathrm{fin}} \to \mathcal{C}$ is any functor, then $i_! F : \mathcal{S} \to \mathcal{C}$ preserves filtered colimits.

We remark that the functor $j_! i : \mathrm{Fun}((\mathcal{S}^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$, obtained by left Kan extending i along the Yoneda embedding of $\mathcal{S}^{\mathrm{fin}}$, has a filtered-colimit preserving right adjoint. This right adjoint is given by the formula

$$X \in \mathcal{S} \longmapsto \mathrm{map}(i(-), X) \in \mathrm{Fun}((\mathcal{S}^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{S})$$

Note that filtered colimits in $\mathrm{Fun}((\mathcal{S}^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{S})$ are computed pointwise so that the above formula shows the right adjoint commutes with filtered colimits precisely because finite spaces are compact in $\mathcal{S}$ by Proposition 1.1.15.

Now note that $i \simeq j_! i \circ j$ by fully-faithfulness of the Yoneda embedding j, so that we can perform the left Kan extension in two steps: first do $j_!$ and then $(j_! i)_!$ which is equivalently given by precomposition along the previous right adjoint. In particular, to conclude it suffices to check that $j_!$ lands in filtered-colimit preserving functors — this follows from Lemma 1.1.4. $\square$

One can do a version of the above adapted to a regular cardinal $\kappa \geq \omega$.

Definition 1.1.20 A category is said to be κ-small if it is given by a simplicial set with a κ-small set of non-degenerate simplices.

The dependence in κ is as follows: if $\kappa \leq \lambda$, then every κ-small category is in particular λ-small.

Definition 1.1.21 A category J is said to be κ-filtered if for every κ-small category $\mathcal{C}$, the functor $\mathrm{cst} : J \to \mathrm{Fun}(\mathcal{C}, J)$ is cofinal.

It follows from the above that the dependency in κ is that if $\kappa \leq \lambda$, then every κ-small filtered category is λ-filtered. In particular, preserving λ-filtered colimits is a *weaker* condition that preserving κ-filtered ones.

Proposition 1.1.22 In $\mathcal{S}$, κ-filtered colimits commute with κ-small limits. Moreover, a category J is κ-filtered if and only if $\operatorname{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves κ-small limits.

Proof. Recall that a functor preserves κ-small limits if it preserves κ-small products and pullbacks. In particular, in light of Proposition 1.1.15, the first claim reduces to proving that κ-filtered colimits commute with κ-small products of spaces.

By Proposition 1.1.15, all the spheres S^n are compact; moreover, $\pi_0 : \mathcal{S} \to \mathrm{Set}$ preserves colimits and arbitrary products[1]. Since κ-filtered colimits are in particular filtered, this reduces the claim to a purely set-theoretical one:

$$\operatorname*{colim}_J \prod_{k\in K} X_{k,j} \longrightarrow \prod_{k\in K} \operatorname*{colim}_J X_{k,j}$$

for $X : J \times K \to \mathrm{Set}$ (with K discrete). This is true and checkable by hand.

In particular, this argument gives one direction of the claimed equivalence and we can run the same proof as in Corollary 1.1.16 (which we will explain later in its correct generality) to get the other implication. □

Definition 1.1.23 An object $X \in \mathcal{S}$ is κ-compact if and only if $\mathrm{Map}(X, -) : \mathcal{S} \to \mathcal{S}$ preserves κ-filtered colimits.

Remark 1.1.24 Unlike in the case $\kappa = \omega$, if κ is uncountable, a space is κ-compact if and only if it is κ-small. The proof starts in the same way: every κ-compact space is a retract of a κ-small space by the same arguments as Corollary 1.1.17, but splitting a retract is a countable colimit since Idem is countably small, hence κ-small spaces are stable under retracts, which concludes.

We also have the following generalization of Proposition 1.1.19:

Proposition 1.1.25 Write $i_\kappa : \mathcal{S}^\kappa$ for the inclusion of full subcategory of κ-small spaces. Suppose $\mathcal{C}$ has κ-filtered colimits, then,

$$i_\kappa^* : \mathrm{Fun}^\kappa(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{S}^\kappa, \mathcal{C})$$

is an equivalence, where the superscript κ denotes the full subcategory of κ-*finitary*, i.e. κ-filtered colimit preserving functors. Its inverse is given by left Kan extension along i.

Proof. We will prove a more general statement in the next section, but the reader is encouraged to adapt the arguments of Proposition 1.1.19. □

The above was a strengthening of the filtered-ness conditions, but one can also weaken the condition of being filtered as follows:

Definition 1.1.26 A category I is *sifted* if the functor $\mathrm{cst} : I \to \mathrm{Fun}(X, I)$ is cofinal for every finite *set* X.

■ **Example 1.1.27** Every filtered category is sifted. It is a well-known fact that $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal (see [Lur18a, Tag 02QP], or play the combinatorial game through Quillen's Theorem A yourself) and since Δ is non-empty, Δ^{op} is sifted. Note that it is not filtered in general. ■

It holds that a functor preserves sifted colimits if and only if it preserves filtered colimits as well as geometric realization, i.e. Δ^{op}-indexed colimits. Note that the situation is different than if one defined this notion in the 1-categorical world; in particular, $\Delta_{\leq 1}$, which models the shape of a reflexive coequalizer, is sifted in the 1-categorical world but not in the higher sense.

Proposition 1.1.28 In $\mathcal{S}$, sifted colimits commute with finite products. Moreover, a category J is sifted if and only if $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ commutes with finite products.

Proof. We already know that filtered colimits commute with finite products, hence it suffices to show that geometric realizations do. As we are not aware of a trick for this, we omit this proof — one strategy is to resolve the geometric realization in a model category of choice and prove it there (see Remark 5.5.8.12 and Lemma 6.1.3.14 of [Lur08]). The other direction of the equivalence will be proven more generally in the next section, and is the same as in Corollary 1.1.16. □

[1] The proof is as follows: show that it holds for Kan complexes, as in this MSE question and then use the fact that the localisation $\mathrm{Kan} \to \mathcal{S}$ preserves small products by virtue of the model structure, in fact only half of it suffices by [Cis19, Proposition 7.7.1]

Usually, an object $X \in \mathcal{C}$ such that $\mathrm{Map}(X, -)$ commutes with sifted colimits is called *compact projective*. In the case of spaces however, there is a much more usual name: *finite sets*.

Proposition 1.1.29 The full subcategory of $\mathcal{S}$ of those X such that $\mathrm{Map}(X, -)$ commutes with sifted colimits is the category FinSet of finite discrete spaces (i.e. sets). Moreover, the inclusion $i : \mathrm{FinSet} \to \mathcal{S}$ induces, for every $\mathcal{C}$ with sifted colimits, an equivalence

$$i^* : \mathrm{Fun}^{\mathrm{sft}}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{FinSet}, \mathcal{C})$$

where the superscript sft denotes the full subcategory of sifted colimit preserving functors.

Proof. The second part will be subsumed in the next section. Let us only prove that FinSet is the claimed category: since $*$ is compact projective, so is every finite coproduct of it by virtue of Proposition 1.1.28.

Conversely, if X is a space such that $\mathrm{Map}(X, -)$ commutes with sifted colimits, we can find a Kan complex model for X itself and therefore realize it as the geometric realization of its n-simplices, which are filtered colimits of finite sets. In particular, there is a sifted colimit of finite sets whose colimit is X.

Therefore, id_X must factor through a finite set and to conclude, we note that the retract of a finite set is necessarily discrete and with finite π_0. □

1.2 Doctrines and colimit-completions

In Proposition 1.1.6, we explain how to freely add colimits to a category. But later throughout the section, we realized that we could also have added less colimits to a bigger category than $*$ and this was still "free" in some sense. In this section, we first explore how to freely add a class of colimits to a category while preserving some or in fact even, forcing a collection of cocones to be colimits. Afterwards, we explore the interaction between adding a shape of colimits freely and adding all colimits while respecting a shape, generalizing Propositions 1.1.19, 1.1.25 and 1.1.29.

Given a collection $S : \{f_\alpha : X_\alpha \to Y_\alpha\}$ of arrows in a category $\mathcal{C}$, we can ask whether a given object Z is S-*local*, i.e. if for every α, the natural map

$$f_\alpha^* : \mathrm{Nat}(Y_\alpha, Z) \longrightarrow \mathrm{Nat}(X_\alpha, Z)$$

is an equivalence. The collection of S-local objects is closed under limits. Note also that we can always saturate a collection of arrows S, i.e. add to S all the morphisms $X_\beta \to Y_\beta$ such that the above precomposition map is an equivalence for S-local objects, and this new collection $\overline{S}$ has the same local objects. Moreover, $\overline{S}$ automatically contains equivalences, is closed under 2-out-of-3 and is closed under colimits; the following is Proposition 6.2.3.12 [Lur18a, Tag 04KG] — we will not reprove it.

Lemma 1.2.1 Suppose that S is a saturated class such that for every $X \in \mathcal{C}$, there is a map $f : X \to Y$ with Y S-local and $f \in S$. Then, the full subcategory $S^{-1}\mathcal{C}$ of S-local objects of $\mathcal{C}$ forms a reflexive subcategory, i.e. the inclusion has a left adjoint $L : \mathcal{C} \to S^{-1}\mathcal{C}$.

Let $\mathcal{K}$ be a collection of "shapes" (i.e. categories) which will serve as indexing our diagrams and that we will typically denote K and fix $\mathcal{C}$ some category. Our goal is to freely adding $\mathcal{K}$-shaped colimits, by this we mean K-colimits for every $K \in \mathcal{K}$ — in fact, we will sometimes need to be a bit more subtle and preserve some colimit diagrams in $\mathcal{C}$.

We let $\mathcal{R} = \{f_\alpha : K_\alpha^\triangleright \to \mathcal{C}\}$ be a collection of diagrams in $\mathcal{C}$, where $K_\alpha \in \mathcal{K}$ and $K_\alpha^\triangleright$ is our notation for freely adding a cocone point to K_α. In human language, we have chosen a collection of $\mathcal{K}$-shaped diagrams in $\mathcal{C}$ *and* a cocone for each of them.

Note the following two points, which are more technicalities than anything: first, we do not require these cocones to be colimit cocones so that we are doing something more general than also preserving some colimits, we are actually enforcing some diagrams to be colimit diagrams. Second, we do require that the K_α are in $\mathcal{K}$ which that if one wants to add say filtered colimits while preserving some cocartesian squares in a category which does not have all pushouts, the resulting category will have all pushouts.

Theorem 1.2.2 There is a category $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ with $\mathcal{K}$-shaped colimits and a functor $\gamma : \mathcal{C} \to \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ which sends every diagram in $\mathcal{R}$ to a colimit diagram. Moreover, for every category $\mathcal{D}$ with $\mathcal{K}$-shaped colimits, precomposition by γ induces an equivalence:

$$\gamma^* : \mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}_{\mathcal{K}}$ denote the full subcategory of functors preserving $\mathcal{K}$-shaped colimits and $\mathrm{Fun}_{\mathcal{R}}$ the full subcategory of those functors sending every diagram in $\mathcal{R}$ to a colimit diagram.

Moreover, if the diagrams in $\mathcal{R}$ are already colimit diagrams in $\mathcal{C}$, γ is fully-faithful.

Proof. We follow essentially the proof of [Lur08, Proposition 5.3.6.2]. Write $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ for the Yoneda embedding and consider the collection S

$$\operatorname*{colim}_{K_\alpha}(j \circ f_\alpha) |_{K_\alpha} \longrightarrow (j \circ f)(*_{K_\alpha}))$$

for each diagram $f_\alpha : K^{\triangleright}_\alpha \to \mathcal{C}$ in $\mathcal{R}$ where $*_{K_\alpha}$ denotes the cocone point of $K^{\triangleright}_\alpha$. We first check the conditions of Lemma 1.2.1. A S-local object ϕ for the above collection is simply a presheaf $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ such that $\phi \circ f_\alpha$ is a limit diagram. We note that given a presheaf ϕ, there is an initial map

$$\eta : \phi \longrightarrow \psi$$

where the right hand side is S-local, and we can simply take ψ to be the limit indexed by the full subcategory $\mathcal{P}(\mathcal{C})_{\phi/}$ spanned by those maps whose target is S-local. But now, for ξ another S-local object, the map

$$\mathrm{Nat}(\psi, \xi) \xrightarrow{\simeq} \mathrm{Nat}(\phi, \xi)$$

is necessarily an equivalence by the universal property of η. The astute reader will (rightfully) complain that the limit defining ψ is not necessarily small and therefore need not exist; the solution to this problem is the small object argument (or the weak Vopěnka principle, which we want to avoid here). We sketch here an argument, due to Manuel Hoff, which explains this fix.

Up to change of universe, we can assume that S is small. We produce a S-equivalence $\phi \to \psi$ (i.e. a map in the saturation of S) such that for every $s \to s'$ in S, there exists a dashed map

$$\begin{array}{ccc} \mathrm{Nat}(s', \phi) & \longrightarrow & \mathrm{Nat}(s', \psi) \\ \downarrow & \swarrow & \downarrow \\ \mathrm{Nat}(s, \phi) & \longrightarrow & \mathrm{Nat}(s, \psi) \end{array}$$

making the square commute. By iterating this construction, the small object argument will provide for us a S-equivalence whose target $\overline{\psi}$ is such that $\overline{\psi} \to *$ has the correct lifting property with respect to maps in S, which is a reformulation of being S-local.

But now, we claim that we can take the map $\psi \to \phi$ be the one given in the pushout square:

$$\begin{array}{ccc} \operatorname*{colim}_{(f: s \to s') \in S} \left(\mathrm{Nat}(s, \phi) \otimes s \coprod_{\mathrm{Nat}(s', \phi) \otimes s} \mathrm{Nat}(s', \phi) \otimes s' \right) & \longrightarrow & \phi \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{(f: s \to s') \in S} \mathrm{Nat}(s, \phi) \otimes s' & \longrightarrow & \psi \end{array}$$

where the tensor denotes the constant colimit indexed by the relevant mapping space. We note that S-equivalences are stable under pushouts against any map, so it suffices to argue that the left hand vertical map is a S-equivalence. Since S-equivalences are closed under colimits, we are reduced to show that:

$$(\mathrm{Nat}(s, \phi) \otimes s) \coprod_{\mathrm{Nat}(s', \phi) \otimes s} (\mathrm{Nat}(s', \phi) \otimes s') \longrightarrow \mathrm{Nat}(s, \phi) \otimes s'$$

is a S-equivalence. By 2-out-of-3 and closure under pushouts, it suffices to check that the maps induced by composition along $f : s \to s'$ in the commutative square which induced the above map are S-equivalences. Now, this last claim follows again because $\mathrm{Nat}(s,\phi)\otimes -$ is a colimit (constant indexed by the left hand space), hence preserves S-equivalences. Finally, this concludes so that we have verified the hypothesis of Lemma 1.2.1.

We let $L : \mathcal{C} \to S^{-1}\mathcal{C}$ be the left adjoint to the category of S-local objects, which exists by Lemma 1.2.1 and we write $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ for the smallest full subcategory of $S^{-1}\mathcal{C}$ containing the image of $L\circ j$ and stable under $\mathcal{K}$-shaped colimits. We claim that for every $\mathcal{D}$ with $\mathcal{K}$-shaped colimits, precomposition along $L\circ j$ induces an equivalence:

$$(L\circ j)^* : \mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}),\mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{R}}(\mathcal{C},\mathcal{D})$$

We first note that $L\circ j$ sends the diagrams of $\mathcal{R}$ to colimits since L inverts the maps in S and preserves colimits, so the above functor is well-defined. Moreover, the minimality hypothesis on $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ implies that this functor is conservative.

Suppose for a moment that $\mathcal{D}$ has all small colimits. This extra-assumption allows the left Kan extension along $L\circ j$ to exists and it is given by $F\mapsto (j_!F)\circ i$ where $i : \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}) \to \mathcal{C}$ is the inclusion. In particular, since i preserves $\mathcal{K}$-shaped colimits, and $j_!F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ all of them by Lemma 1.1.4, the left Kan extension $(L\circ j)_!$ does restrict to the wanted categories, so that it is a left adjoint to $(L\circ j)^*$.

Now given $F : \mathcal{C}\to\mathcal{D}$ that sends $\mathcal{R}$ to colimits, we want to check that the map

$$F \longrightarrow (L\circ j)^*(L\circ j)_!F$$

is an equivalence. Now note that because j is fully-faithful, it will suffice to prove that $j_!F \to L^*L_!j_!F$ is an equivalence. But because F sends every f_α to a colimit already, the functor $j_!F$ factors through the category of S-local objects $S^{-1}\mathcal{P}(\mathcal{C})$ (in fact this is an if and only if), and therefore is canonically equivalent to $L_*L_!j_!F$.

Finally, we reduce to the case where $\mathcal{D}$ has small colimits. Note that $\mathcal{D}\to\overline{\mathcal{D}} := \mathrm{Fun}(\mathcal{D},\mathcal{S})^{\mathrm{op}}$ is colimit-preserving, as the opposite of the Yoneda embedding of $\mathcal{D}^{\mathrm{op}}$, and its target has all small colimits since $\mathcal{S}$ has small limits. Hence, the above applies to $\overline{\mathcal{D}}$ and it suffices to show that the following square is cartesian:

$$\begin{array}{ccc}
\mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}),\mathcal{D}) & \longrightarrow & \mathrm{Fun}_{\mathcal{R}}(\mathcal{C},\mathcal{D}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}),\overline{\mathcal{D}}) & \longrightarrow & \mathrm{Fun}_{\mathcal{R}}(\mathcal{C},\overline{\mathcal{D}})
\end{array}$$

In turn, this means showing that if $F : \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}) \to \overline{\mathcal{D}}$ preserves $\mathcal{K}$-shaped colimits and restricts to $\mathcal{D}$ along α, then F itself was already landing in $\mathcal{D}$. But the full subcategory $F^{-1}(\mathcal{D})$ contains $\mathcal{C}$ by assumption and is closed under $\mathcal{K}$-shaped colimits, hence $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})\subset F^{-1}(\mathcal{D})$ which concludes by minimality.

It remains to explain the last claim of the Theorem: suppose that every diagram in $\mathcal{R}$ is a colimit diagram. Then, to show that α is fully-faithful, it suffices to check that $j : \mathcal{C}\to\mathcal{P}(\mathcal{C})$ lands in S-local objects for the aforementioned collection of maps S. We have to show for every $X\in\mathcal{C}$

$$\mathrm{Nat}(j(F(*_R)), j(X)) \xrightarrow{\simeq} \mathrm{Nat}(\operatorname*{colim}_R j\circ F\mid_R, j(X))$$

for every diagram $F : \overline{R}\to\mathcal{C}$. Using the Yoneda lemma and the fact that $F(*_R)\simeq \mathrm{colim}_R F\mid_R$, this is clearly follows from the fact that

$$\mathrm{Map}(\operatorname*{colim}_{r\in R} F(r), X) \longrightarrow \lim_{r\in R}\mathrm{Map}(F(r),X)$$

is an equivalence. □

We say that $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ is $\mathcal{C}$ to which we have added $\mathcal{K}$-shaped colimits while forcing $\mathcal{R}$ to be colimits. If $\mathcal{R}$ is already a collection of colimits cocones, we say that instead "while *preserving* $\mathcal{R}$-colimits". Finally, if $\mathcal{R}=\emptyset$, we say that we have *freely* added $\mathcal{K}$-shaped colimits to $\mathcal{C}$

■ **Example 1.2.3** If $\mathcal{K} = \mathbf{Cat}$ and $\mathcal{R} = \emptyset$, Proposition 1.1.6 has guaranteed that $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}) \simeq \mathcal{P}(\mathcal{C})$ and j is the Yoneda embedding.

We have also seen that $\mathcal{S}$ is the common value of $\mathcal{P}^{\omega-filt}_{\emptyset}(\mathcal{S}^{\mathrm{fin}})$, $\mathcal{P}^{\kappa-filt}_{\emptyset}(\mathcal{S}^{\kappa})$ and $\mathcal{P}^{sifted}_{\emptyset}(\mathrm{FinSet})$. In all three cases, we also got that the resulting category had all small colimits and the inclusion preserved the complementary type of them, i.e.

$$\mathcal{P}^{\omega-filt}_{\emptyset}(\mathcal{S}^{\mathrm{fin}}) \simeq \mathcal{P}^{small}_{fin}(\mathcal{S}^{\mathrm{fin}})$$
$$\mathcal{P}^{\kappa-filt}_{\emptyset}(\mathcal{S}^{\kappa}) \simeq \mathcal{P}^{small}_{\kappa-small}(\mathcal{S}^{\kappa})$$
$$\mathcal{P}^{sifted}_{\emptyset}(\mathrm{FinSet}) \simeq \mathcal{P}^{small}_{\coprod}(\mathrm{FinSet})$$

where we hope all of the super/subscripts are clear. This is not a coincidence, and we will explain it later in the section. ■

Remark 1.2.4 Given $\mathcal{K}$ and a choice $\mathcal{R}$ of cocones in $\mathcal{C}$, the association $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(-)$ naturally refines to a functor thanks to its universal property. Its source is the category of pairs $(\mathcal{C}, \mathcal{R})$ where maps are functors which preserve the collections of the chosen cocones[a]; its target is the category of categories with $\mathcal{K}$-shaped colimits.

In fact, $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(-)$ is a left adjoint to the functor sending $\mathcal{D}$ with $\mathcal{K}$-shaped colimits to $(\mathcal{D}, \{\mathcal{K}-$shaped colimits cocones in $\mathcal{D}\}$. This is especially practical because this adjunction often descends when considering subcategories of the form $(\mathcal{C}, \mathcal{R})$ with $\mathcal{R}$ chosen functorially (such as finite colimit cocones, etc ...).

[a] More precisely, $f : \mathcal{C} \to \mathcal{D}$ must be such that if $p : K^{\triangleright}_{\alpha} \to \mathcal{C}$ is in $\mathcal{R}_{\mathcal{C}}$, then $f \circ p$ is $\mathcal{R}_{\mathcal{D}}$

We now showcase three examples which will be important for us later on. The first one will be ubiquitous throughout this lecture.

Definition 1.2.5 The *κ-inductive completion* of $\mathcal{C}$, denoted $\mathrm{Ind}_{\kappa}(\mathcal{C})$, is the category obtained from $\mathcal{C}$ by freely adding κ-filtered colimits, i.e. $\mathrm{Ind}_{\kappa}(\mathcal{C}) \simeq \mathcal{P}^{\kappa-filt}_{\emptyset}(\mathcal{C})$.

The second kind of example will be a little less common for us, but does have its use in other parts of higher category theory:

Definition 1.2.6 The *non-abelian derived category* of $\mathcal{C}$, denoted $\mathcal{P}_{\Sigma}(\mathcal{C})$, is the category obtained from $\mathcal{C}$ by freely adding sifted colimits, i.e. $\mathcal{P}_{\Sigma}(\mathcal{C}) \simeq \mathcal{P}^{sifted}_{\emptyset}(\mathcal{C})$. In recent years, this category has also been known as the *animation* of $\mathcal{C}$.

We write Ret for the category generated by the graph two vertices A, X, with non-trivial maps $i : A \to X$, $r : X \to A$ such that $r \circ i = \mathrm{id}$. We write Idem for the full subcategory spanned by the vertex X. We note that the inclusion Idem $\to$ Ret is cofinal, hence every functor with source Ret is left Kan extended from Idem. Therefore, a functor $F : \mathrm{Idem} \to \mathcal{C}$ admits a colimit if and only if it extends to Ret.

Definition 1.2.7 We write $\mathrm{Idem}(\mathcal{C})$ for the category obtained from $\mathcal{C}$ by freely adding retracts to idempotent, i.e. $\mathrm{Idem}(\mathcal{C}) \simeq \mathcal{P}^{\mathrm{Idem}}_{\emptyset}(\mathcal{C})$.

Remark 1.2.8 If $\mathcal{C}$ admits finite colimits, it need not be that every idempotent has a colimit, i.e. Idem is *not* a finite category in the higher categorical world.

We finish this section by ideas of Rezk [Rez21], himself based on 1-categorical results notably by Adámek, Lawvere, Rosický and many of their collaborators, from which tries to encapsulate the fact that the three Proposition 1.1.15 1.1.22, 1.1.28 have similar statements and similar proofs (which we ourselves have skipped for this very reason).

For the rest of the section, fix $\mathcal{U}$ a collection of small categories which we call our *doctrine*.

Definition 1.2.9 A category J is called *$\mathcal{U}$-filtered* if $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves $\mathcal{U}$-shaped limits.

A category J is called *weakly $\mathcal{U}$-filtered* if for every $U \in \mathcal{U}$, the functor $\mathrm{cst} : J \to \mathrm{Fun}(U^{\mathrm{op}}, J)$

is cofinal.

■ **Example 1.2.10** If J has all U^{op}-colimits for $U \in \mathcal{U}$, then J is weakly $\mathcal{U}$-filtered, since cst has a left adjoint. Note also that although it looks different, this definition recovers the one for κ-filtered using $\mathcal{U} = \{\kappa\text{-small categories}\}$ since those are closed under op. ■

The reason for the op appearing in our definition is so that they disappear in the following:

Lemma 1.2.11 A category J is weakly $\mathcal{U}$-filtered if and only if $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves $\mathcal{U}$-shaped limits of corepresentable functors. In particular, $\mathcal{U}$-filtered categories are weakly $\mathcal{U}$-filtered.

Proof. By Quillen's Theorem A [Lur08, Theorem 4.1.3.1], J is weakly $\mathcal{U}$-filtered if and only if for every $U \in \mathcal{U}$ and every $F : U^{\mathrm{op}} \to J$, the category $\mathrm{Fun}(U^{\mathrm{op}}, J)_{F/} \times_{\mathrm{Fun}(U^{\mathrm{op}}, J)} J$ is contractible. We recall that this category is the total space of the unstraightening of $\mathrm{Nat}(F, \mathrm{cst}(j)) \simeq \lim_{u \in U} \mathrm{Map}(F(u), j)$ as a functor $\mathcal{J} \to \mathcal{S}$.

On the other hand, colim_J preserves U-limits of corepresentables if for every $F : U^{\mathrm{op}} \to J$ the map

$$\eta : \mathop{\mathrm{colim}}_{j \in J} \lim_{u \in U} \mathrm{Map}(F(u), j) \longrightarrow \lim_{u \in U} \mathop{\mathrm{colim}}_{j \in J} \mathrm{Map}(F(u), j)$$

is an equivalence.

Note that by Lemma 1.1.2 $\mathrm{colim}_{j \in J} \mathrm{Map}(F(u), j)$ is always contractible as $\mathrm{Map}(F(u), -)$ is classified by $J_{F(u)/} \to J$ whose total category has an initial object. In particular, if η is an equivalence then the left hand side is contractible and therefore again by Lemma 1.1.2, the category $\mathrm{Fun}(U^{\mathrm{op}}, J)_{F/} \times_{\mathrm{Fun}(U^{\mathrm{op}}, J)} J$ is weakly contractible. Reciprocally, if this category is weakly-contractible, then the left hand side is also contractible hence both sides are contractible and therefore the map is an equivalence. □

Warning 1.2.12 The converse need not hold. Rezk has some examples in section 6 of his paper. Let us also mention another. Consider $\mathcal{U} := \{\kappa\text{-small sets}\}$ for some $\kappa > \omega$, then we claim that the category Δ_κ of linearly order κ-small sets and order preserving maps is weakly $\mathcal{U}$-filtered; the argument adapts from the standard argument showing that Δ is (weakly) ω-sifted.

Nonetheless, it can be shown that κ-small products of spaces do not commute with Δ_κ-indexed colimits. In fact, the main result of [AKV00] shows that in the 1-categorical world, $\mathcal{U}$-filtered categories are actually κ-filtered (!), which Δ_κ is not.

Therefore, if J is such that $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves κ-small products, so will the colimit functor valued in Set since π_0 preserves colimits and arbitrary products[a], and therefore J will be κ-filtered. Since κ-small sets are in particular κ-small categories, we see that colim_J preserves κ-small products if and only if J is κ-filtered.

[a] Yet another thing we do not know of a good reference for, except for `https://math.stackexchange.com/questions/713792/does-pi-0-preserve-infinite-products`this MSE answer which is not quite written in a model-independent way.

Definition 1.2.13 We call a doctrine *sound* if the converse of Lemma 1.2.11 holds.

■ **Example 1.2.14** The doctrines of κ-small categories are sound for every κ regular, as we have checked in Proposition 1.1.22; since every small category is κ-small for some κ, this extends to the doctrine of all small categories. Note that this fails for κ-small sets if $\kappa > \omega$ by the above. ■

We will not draw an explicit criterion for soundness in this version of the course notes, but we want to include one eventually.

Given a doctrine $\mathcal{U}$, we write $\overline{\mathcal{U}}$ for the collection of U such that $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-shaped limits for every $\mathcal{U}$-filtered J. By definition $\mathcal{U} \subset \overline{\mathcal{U}}$. Moreover, if $\mathcal{U} \subset \mathcal{V}$, then $\overline{\mathcal{U}} \subset \overline{\mathcal{V}}$.

Definition 1.2.15 A doctrine $\mathcal{U}$ is *regular* if $\mathcal{U} = \overline{\mathcal{U}}$.

■ **Example 1.2.16** The doctrine of κ-small categories is regular as soon as κ is a regular cardinal. ■

The categories in $\overline{\emptyset}$ are often known are universal (co)limits. In particular, for every regular doctrine $\mathcal{U}$, we have $\overline{\emptyset} \subset \mathcal{U}$.

Lemma 1.2.17 The category Idem belongs to $\overline{\emptyset}$, i.e. it is preserved by $\operatorname{colim}_J : \operatorname{Fun}(J, \mathcal{S}) \to \mathcal{S}$ for every J.

Proof. This follows directly from the fact that $\mathrm{Idem} \to \mathcal{C}$ has a colimit if and only if it extends to Ret. □

Note that since $\mathrm{Idem}^{\mathrm{op}} \simeq \mathrm{Idem}$, the above statement also holds for limit preservation.

Definition 1.2.18 An object $X \in \mathcal{C}$ is $\mathcal{U}$-compact if $\operatorname{Map}(X, -) : \mathcal{C} \to \mathcal{S}$ commutes with $\mathcal{U}$-filtered colimits.

■ **Example 1.2.19** The point $* \in \mathcal{S}$ is $\mathcal{U}$-compact for any collection $\mathcal{U}$. Any initial object is $\mathcal{U}$-compact for any $\mathcal{U}$. ■

Note that $\mathcal{U}$-compactness only depends on the class of $\mathcal{U}$-filtered categories, which itself only depends on the regular doctrine $\overline{\mathcal{U}}$ generated by $\mathcal{U}$.

Lemma 1.2.20 Let $\mathcal{C}$ be a category. The subcategory of $\mathcal{U}$-compact objects of $\mathcal{C}$ is closed under all the U^{op}-indexed colimits that exist, for $U \in \overline{\mathcal{U}}$. In particular, $\mathcal{U}$-compact objects are always closed under retracts.

Proof. By Lemma 1.2.17, $\mathrm{Idem} \in \overline{\emptyset} \subset \overline{\mathcal{U}}$. Hence, it suffices to prove the first claim. Given a U^{op}-indexed diagram of compact objects X_u which admits a colimit X in $\mathcal{C}$, we have $\operatorname{Map}(X, -) \simeq \lim_{u \in U} \operatorname{Map}(X_u, -)$. Now given J which is $\mathcal{U}$-filtered and a J-indexed diagram Y_j, the canonical map

$$\operatorname{colim}_{j \in J} \lim_{u \in U} \operatorname{Map}(X_u, Y_j) \xrightarrow{\simeq} \lim_{u \in U} \operatorname{colim}_{j \in J} \operatorname{Map}(X_u, Y_j)$$

is an equivalence. Using that each X_u is compact, we can pull the colimit on the right hand side inside, i.e. this term identifies with $\operatorname{Map}(X, \operatorname{colim}_j Y_j)$ whereas the left hand side is $\operatorname{colim}_j \operatorname{Map}(X, Y_j)$ which concludes. □

Another consequence of the above is that the full subcategory of $\mathcal{S}$ of $\mathcal{U}$-compact objects always contain the full subcategory generated by $*$ under U^{op}-indexed colimits for $U \in \overline{\mathcal{U}}$.

Remark 1.2.21 Actually, the previous proof only used that colim_J commutes with corepresentable presheaves, so it also applies to a version of compact objects defined with respect to weakly $\mathcal{U}$-filtered colimits.

Definition 1.2.22 Let $\mathcal{C}$ be a small category. We write $\operatorname{Ind}_{\mathcal{U}}(\mathcal{C})$ for the category obtained from $\mathcal{C}$ by freely $\mathcal{U}$-filtered colimits, i.e. $\operatorname{Ind}_{\mathcal{U}}(\mathcal{C}) \simeq \mathcal{P}_{\emptyset}^{\mathcal{U}-\mathrm{filt}}(\mathcal{C})$ in the notations of Theorem 1.2.2.

■ **Example 1.2.23** It is standard to write $\operatorname{Ind}(\mathcal{C})$ for $\operatorname{Ind}_{\omega-filt}(\mathcal{C})$, i.e. freely adding filtered colimits to a category, and call it the *inductive completion of* $\mathcal{C}$ (in the sense of objects in $\operatorname{Ind}(\mathcal{C})$ being formal inductive systems, inductive being a old (possibly weaker?) name for filtered.

More generally, we will write $\operatorname{Ind}_\kappa(\mathcal{C})$ for freely adding κ-filtered colimits to a category. ■

■ **Example 1.2.24** The "old-school" name and notation for $\operatorname{Ind}_{sifted}(\mathcal{C})$, freely adding sifted colimits (equivalently, filtered colimits and geometric realizations) is $\mathcal{P}_\Sigma$, the *non-abelian derived category*. If $\mathcal{C}$ has finite coproducts, the following theorem will show that it also coincides with finite-product preserving presheaves on spaces, a process often call *animation* under the influence of the condensed mathematics crowd, which call $\mathcal{S}$ the category of anima.

In particular, as a consequence of what we explained in the previous section, $\mathcal{S}$ is the animation of the finite coproduct closure of $*$, i.e. the category FinSet of finite sets. ■

Theorem 1.2.25 — Rezk. Let $\mathcal{U}$ be a sound doctrine. Suppose $\mathcal{C}$ is a category with U^{op}-colimits for every $U \in \mathcal{U}$, then there is an equivalence

$$\mathrm{Ind}_{\mathcal{U}}(\mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

where $\mathrm{Fun}_{\mathcal{U}-\lim}$ designates the full subcategory of $\mathcal{U}$-limit preserving presheaves.

Moreover, $\mathrm{Ind}_{\mathcal{U}}(\mathcal{C})$ has U^{op}-colimits for every $U \in \mathcal{U}$ and they are preserved by the fully-faithful $j_{\mathcal{U}} : \mathcal{C} \to \mathrm{Ind}_{\mathcal{U}}(\mathcal{C})$, so there is another equivalence $\mathrm{Ind}_{\mathcal{U}}(\mathcal{C}) \simeq \mathcal{P}^{\mathcal{U}-\mathrm{filt}}_{\mathcal{U}^{\mathrm{op}}}(\mathcal{C})$.

Proof. We begin by reducing the second assertion to the first; for this, we prove that the category $\mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ has U^{op}-colimits for $U \in \mathcal{U}$ and they are preserved by the Yoneda embedding. It follows from Lemma 1.2.1 that the inclusion

$$\mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \longrightarrow \mathcal{P}(\mathcal{C})$$

has a left adjoint. Indeed, by the Yoneda lemma, we see that the left hand side is the category of presheaves which are local with respect to the collection of maps

$$\operatorname*{colim}_{u \in U^{\mathrm{op}}} j(X^{\mathrm{op}}(u)) \longrightarrow j(\operatorname*{colim}_{u \in U^{\mathrm{op}}} X^{\mathrm{op}}(u))$$

for every $X : U \to \mathcal{C}^{\mathrm{op}}$, and this collection satisfies the hypotheses of 1.2.1. In particular, since $\mathcal{P}(\mathcal{C})$ has colimits, so does $\mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and the above collection of maps makes it clear that U^{op}-colimits are preserved by j.

We now check that $\mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ has the wanted universal property. In fact, we check that it coincides with the description of $\mathcal{P}^{\mathcal{U}-\mathrm{filt}}_{\emptyset}(\mathcal{C})$ given by Theorem 1.2.2. Note that since our $\mathcal{R}$ is empty, this is precisely the smallest full subcategory of $\mathcal{P}(\mathcal{C})$ containing the image of the Yoneda and closed under $\mathcal{U}$-filtered colimits.

We first remark that if $X \in \mathcal{C}$, then $j(X) := \mathrm{Map}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ preserves $\mathcal{U}$-limits. Moreover, $\mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is closed under $\mathcal{U}$-filtered colimits in $\mathcal{P}(\mathcal{C})$ precisely because those commute with $\mathcal{U}$-limits in spaces. To conclude, it suffices to show that every presheaf $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which commute with $\mathcal{U}$-limits is a $\mathcal{U}$-filtered colimit of representables.

We have already proven this statement: indeed, recall from the proof of Proposition 1.1.6 that the canonical map

$$\operatorname*{colim}_{j(Y) \to \phi} \mathrm{Map}(X, Y) \longrightarrow \phi(X)$$

is an equivalence. We claim that the category $\mathrm{Un}^{cart}(\phi) \simeq \mathcal{P}(\mathcal{C})_{/\phi} \times_{\mathcal{P}(\mathcal{C})} \mathcal{C}$ which indexes the colimit is $\mathcal{U}$-filtered, i.e. that space-valued colimits indexed by $\mathrm{Un}^{cart}(\phi)$ commute with $\mathcal{U}$-limits. Note that it is automatically weakly $\mathcal{U}$-filtered since it admits U^{op}-colimits; this is straightforward from identifying $\mathrm{Un}^{cart}(\phi) \simeq \mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\phi} \times_{\mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})} \mathcal{C}$ and the first statement we proved. □

Remark 1.2.26 We found throughout the proof that another description of $\mathrm{Ind}_{\mathcal{U}}(\mathcal{C})$ is the category of presheaves ϕ such that $\mathrm{Un}(\phi)$ is a $\mathcal{U}$-filtered category. In fact, the above shows that this description stands even if $\mathcal{C}$ has not enough $\mathcal{U}^{\mathrm{op}}$-shaped colimits and $\mathcal{U}$ is not necessarily sound. We refer to Rezk's manuscript [Rez21] for a more general picture (including what happens when $\mathcal{U}$ is not sound).

1.3 Accessible and presentable categories

Recall that we write $\mathrm{Ind}_\kappa(\mathcal{C})$ for the κ-filtered colimit completion of $\mathcal{C}$.

Definition 1.3.1 A category $\mathcal{C}$ is *κ-accessible* for a regular cardinal κ. if there exist a small $\mathcal{C}_0$ and an equivalence $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0)$. It is accessible if it is κ-accessible for some κ.

A functor $F : \mathcal{C} \to \mathcal{D}$ between κ-accessible categories is κ-accessible if it preserves κ-filtered colimits.

In particular, κ-accessible categories have κ-filtered colimits. Moreover, every object is a κ-filtered colimit of small objects. In fact, this characterizes them:

Proposition 1.3.2 A category $\mathcal{C}$ is κ-accessible if and only if if there is a set of κ-compact objects S such that the smallest category closed under κ-filtered colimits containing S is all of $\mathcal{C}$.

Proof. This is straightforward. □

Let us now run through inheritance properties, which we will not prove (they are listed in the order of appearance of §5.4 in [Lur08]).

Proposition 1.3.3 Suppose $\mathcal{C}$ is accessible and K a small category, then

1. $\mathrm{Fun}(K, \mathcal{C})$ is accessible.
2. If $p : K \to \mathcal{C}$ is a functor, $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ are accessible.
3. If $\mathcal{C} \to \mathcal{D} \leftarrow \mathcal{C}'$ is a span of accessible functors between accessible categories, so is the pullback $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}'$ as well as the projections.
4. Small coproducts of accessible categories are accessible.
5. Accessible categories are closed under small products in **Cat**, thus small limits thanks to the previous points.
6. If $p : K \to \mathbf{CAT}$ takes values in the subcategory of accessible categories and accessible functors, then both

$$\mathrm{lax}\lim(p) := \mathrm{Fun}_{/K}(K, \mathrm{Un}^{cart}(p)) \qquad \mathrm{oplax}\lim(p) := \mathrm{Fun}_{/K}(K, \mathrm{Un}(p))$$

are accessible.

Definition 1.3.4 A category $\mathcal{C}$ is *presentable* if it is accessible and admits small colimits. A presentable category $\mathcal{C}$ is *κ-compactly generated* if it is κ-accessible.

■ **Example 1.3.5** The category $\mathcal{S}$ is presentable, and so are all the functor categories $\mathrm{Fun}(K, \mathcal{S})$ by Proposition 1.3.3 and the fact that colimits are computed pointwise. ■

By Theorem 1.2.25, if $\mathcal{C}$ is a small category with κ-small colimits, then $\mathrm{Ind}_\kappa(\mathcal{C})$ has all small colimits and by definition is accessible, hence is presentable. The converse is true, as we explain, and there is a canonical, maximal candidate for any presentable category: its full subcategory of compact objects.

Lemma 1.3.6 Suppose $\mathcal{C}$ is κ-compactly generated, then $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$, where $\mathcal{C}^\kappa$ denotes the full subcategory of κ-compact objects in $\mathcal{C}$.

Proof. Note that $\mathcal{C}^\kappa$ has κ-small colimits by Lemma 1.2.20. The universal property provides a colimit-preserving map $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{C}$ which is the identity on $\mathcal{C}^\kappa$. By Proposition 1.3.2, both categories are generated under κ-filtered colimits by $\mathcal{C}^\kappa$ so this functor is essentially surjective but it is also fully-faithful by virtue of the explicit formula for mapping spaces in $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$. □

■ **Example 1.3.7** We have seen that $\mathcal{S}$ is compactly-generated (i.e. $\kappa = \omega$) in Proposition 1.1.5. ■

Theorem 1.3.8 — Simpson. A category is presentable $\mathcal{D}$ if and only if it arises as an accessible left Bousfield localisation $L : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ of a presheaf category.

Proof. Here, we use the term Bousfield localisation to stress that we want the localisation to have a right adjoint which is necessarily fully-faithful, and the accessibility is the condition that the right adjoint has to be accessible. It is clear that accessible left Bousfield localisation of presheaves have colimits, computed by the formula $L(\mathrm{colim}\, R(X_\alpha))$, hence are presentable.

Reciprocally, fix κ such that $\mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{D}^\kappa)$. Note that $\mathcal{D}^\kappa$ admits κ-small limits, therefore by Theorem 1.2.25, we get that

$$\mathrm{Ind}_\kappa(\mathcal{D}^\kappa) \simeq \mathrm{Fun}^{\kappa-\lim}((\mathcal{D}^\kappa)^{\mathrm{op}}, \mathcal{S})$$

But the right hand side is the full subcategory of $\mathcal{P}(\mathcal{D}^\kappa)$ spanned by those objects which are local with respect to

$$\operatorname*{colim}_{x\in K} j(f(x)) \longrightarrow j(\operatorname*{colim}_{x\in K} f(x))$$

where $f : K :\to \mathcal{D}^\kappa$ ranges over κ-small diagrams. This concludes. □

Let us also give the following criterion for κ-compact generation, which was a folklore result that first appeared in this version in [CDH+23] to our knowledge:

Proposition 1.3.9 Let S be a set of κ-compact objects in a cocomplete category $\mathcal{C}$. Suppose S jointly detects equivalences, i.e. $f : X \to Y$ is an equivalence if and only if for every $s \in S$, the map

$$f_* : \mathrm{Map}(s, X) \longrightarrow \mathrm{Map}(s, Y)$$

is an equivalence. Then, $\mathcal{C}$ is κ-compactly generated.

Proof. Denote $\mathcal{C}_S$ the smallest full subcategory of $\mathcal{C}$ closed under κ-small colimits of objects S.

There is a colimit-preserving, fully-faithful $\alpha : \mathrm{Ind}_\kappa(\mathcal{C}_S) \to \mathcal{C}$ and to check it is an equivalence, it suffices to see that $\mathrm{id}_{\mathcal{C}}$ is left Kan extended from its restriction along $\mathcal{C}_S$, i.e. that for every $X \in \mathcal{C}$, the map

$$\operatorname*{colim}_{(t\to X)\in \mathcal{C}_S\times_{\mathcal{C}}\mathcal{C}_{/X}} t \longrightarrow X$$

is an equivalence. Note that since it has κ-small colimits, $\mathcal{C}_S$ is κ-filtered and therefore, this also holds for $\mathcal{C}_S \times_{\mathcal{C}} \mathcal{C}_{/X}$. In consequence, using that every object of $\mathcal{C}_S$ is κ-compact, it suffices to show that the natural map

$$\operatorname*{colim}_{(t\to X)\in \mathcal{C}_S\times_{\mathcal{C}}\mathcal{C}_{/X}} \mathrm{Map}(-, t) \longrightarrow \mathrm{Map}(-, X)$$

of $\mathcal{C}_S$-presheaves is an equivalence. We have already established this formula in Proposition 1.1.6, up to the Yoneda lemma to transform $\mathrm{Nat}(j(t), j(X))$ into $\mathrm{Map}_{\mathcal{C}}(t, X)$. □

Corollary 1.3.10 The singleton $\{[1] = \{0 \to 1\}\}$ detects equivalences in **Cat**, the category of small categories. Hence **Cat** is compactly-generated (and by one point!).

When further restricting to $\mathbf{Cat}^{\mathrm{Ex}}$, the situation is even simpler: the singleton $\{\mathrm{Sp}^{\mathrm{fin}}\}$ detects equivalences in $\mathbf{Cat}^{\mathrm{Ex}}$. Hence $\mathbf{Cat}^{\mathrm{Ex}}$ is also compactly-generated by one point, and it is its unit for the canonical symmetric structure (see 1.4.7 and the discussion that follows).

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $\mathrm{Ar}(F)^{\simeq} : \mathrm{Ar}(\mathcal{C})^{\simeq} \to \mathrm{Ar}(\mathcal{D})^{\simeq}$ is an equivalence of underlying groupoids. In particular, since this functor is surjective, for every $Y \in \mathcal{D}$, there is an equivalence $\alpha : X \to X'$ which maps to id_Y. In particular, $F(X) = Y$ so that F is surjective hence it suffices to check that F is fully-faithful to get the first point.

By assumption, for every $f : X \to Y, g : Z \to T$, F induces an equivalence

$$\mathrm{Iso}_{\mathrm{Ar}(\mathcal{C})}(f, g) \longrightarrow \mathrm{Iso}_{\mathrm{Ar}(\mathcal{D})}(F(f), F(g))$$

between the subspaces of equivalences in the mapping spaces in the respective arrow categories. But note that $\mathrm{Map}(X, Y)$ is a retract of $\mathrm{colim}_{f\in\mathrm{Map}(X,Y)} \mathrm{Iso}_{\mathrm{Ar}(\mathcal{C})}(f, f)$ since there is a map

$$\operatorname*{colim}_{f\in\mathrm{Map}(X,Y)} \mathrm{Iso}_{\mathrm{Ar}(\mathcal{C})}(f, f) \longrightarrow \mathrm{Map}(X, Y)$$

projecting on the first component which has a section which informally sends $f : X \to Y$ to the square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \| \\ X & \xrightarrow{f} & Y \end{array}$$

In consequence, the map

$$\mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(F(X), F(Y))$$

is a retract of an equivalence hence an equivalence itself. This gives the first part

We can play a similar kind of trick with even less assumptions if we suppose that $F : \mathcal{C} \to \mathcal{D}$ is exact between stable categories (in fact product-preserving between additive categories suffices). If $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is an equivalence, then F is surjective.

This time, we use that the projection $\mathrm{Iso}(X \oplus Y, X \oplus Y) \to \mathrm{Map}(X, Y)$, obtained by postcomposing by $X \oplus Y \to Y$ and precomposing by $X \to X \oplus Y$, has a section which sends f to

$$f \in \mathrm{Map}(X, Y) \longmapsto \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \in \mathrm{Iso}(X \oplus Y, X \oplus Y)$$

We also get that our wanted map is a retraction of an equivalence, hence an equivalence itself. This concludes. □

Remark 1.3.11 After writing the above, we learned from [Aok25, Lemma 2.4] of the following different proof, based on Proposition 1.1.19, which is too pretty not to share. Viewing categories as complete Segal spaces produces a fully-faithful functor **Cat** $\to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ sending $\mathcal{C}$ to $\mathrm{Fun}([\bullet], \mathcal{C})^{\simeq}$. This functor has a left adjoint by Lemma 1.2.1 since being a (complete) Segal object is being local with respect to inclusion of the spines $[1] \times_{[} 0] \dots \times_{[} 0][1] \to [n]$.

Note that the inclusion preserves filtered colimits as each $[n]$ is compact in **Cat**, hence the left adjoint preserves compact objects. In particular, it follows that **Cat** is generated under colimits by the image of the compact generators of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ which, by Proposition 1.1.19, are the $[n]$. But $[0]$ is a retract of $[1]$ and $[n]$ is in a finite colimit of $[1]$ hence we are done.

We now turn to one of the nicest feature of presentable categories, namely the adjoint functor theorem. For this, we will first need the following, which we essentially already proved in Theorem 1.2.25.

Proposition 1.3.12 Let $\mathcal{C}$ be a presentable category and $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ a presheaf. Then, F is representable if and only if F preserves small limits.

Proof. Representable functors preserve limits (without hypotheses on $\mathcal{C}$), we now work for the converse. By definition, F is representable if and only if it lies in the image of $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$, hence we have to show that if $\mathcal{C}$ is presentable,

$$\mathcal{C} \longrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

is an equivalence. By assumption, $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ so that by a game of ops, the right hand side also identifies with the category $\mathrm{Fun}^{\kappa-lim}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S})$ of κ-small limit preserving functors out of $(\mathcal{C}^\kappa)^{\mathrm{op}}$

But now, we can appeal to Theorem 1.2.25 (see Example 1.2.14 for the soundness claim): $\mathcal{C}^\kappa$ has κ-small colimits by presentability so that we have an identification

$$\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \simeq \mathrm{Fun}^{\kappa-\mathrm{lim}}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S})$$

induced by the Yoneda embedding. Combining the two results concludes. □

In particular, we have shown $\mathcal{C} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, so we get immediately:

Corollary 1.3.13 Presentable categories admit all small limits.

The dual situation is slightly more complicated. We would like to argue with opposites everywhere but $\mathcal{C}^{\mathrm{op}}$ need not be presentable if $\mathcal{C}$ is. However, note that if $\mathcal{C}$ is κ-compactly generated, then $\mathcal{C}^\kappa$ is closed under κ-small limits in $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$, since in the latter κ-small limits are computed pointwise in $\mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S})$ where it is clear that $\mathcal{C}^\kappa$ is closed under limits.

We write $\mathrm{Pro}_\kappa(\mathcal{C}) := \mathrm{Ind}_\kappa(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and we note that $\mathcal{C} \to \mathrm{Pro}_\kappa(\mathcal{C})$ exhibits its target as the free completion of $\mathcal{C}$ by κ-cofiltered limits.

Proposition 1.3.14 Let $\mathcal{C}$ be a presentable category and $F : \mathcal{C} \to \mathcal{S}$ a copresheaf. Then, F is corepresentable if and only if F preserves small limits and is accessible.

Proof. If $F \simeq \mathrm{Map}(X, -)$ for some $X \in \mathcal{C}$ then it preserves limits and since $\mathcal{C}$ is presentable, X is necessarily κ-compact for some κ hence F is also κ-accessible.

Reciprocally, fix κ such that both F is κ-accessible and $\mathcal{C}$ is κ-compactly generated. We note that the restriction $\widehat{F} : \mathcal{C}^\kappa \to \mathcal{S}$ preserves κ-small limits by the above and it suffices to check it is corepresented. We let $\mathrm{Un}(F) \to \mathcal{C}$ be the left fibration classifying F and consider the object of $\mathcal{C}$, which exists by Corollary 1.3.13,

$$X := \lim_{\substack{(x,\alpha)\in \mathrm{Un}(F) \\ x \in \mathcal{C}^\kappa}} x$$

of $\mathrm{Un}(F)$ restricted to those objects which are underlying κ-compact. Differently stated, this is the limit in $\mathcal{C}$ of the canonical functor obtained by pulling back $\mathrm{Un}(F)$ over $\mathcal{C}^\kappa$.

Since F preserves limits, the collection of $(x, \alpha) \in \mathrm{Un}(F)$ with $x \in \mathcal{C}^\kappa$ determines a point $\overline{\alpha}$ in the limit $F(X)$. Therefore the Yoneda embedding provides a map:

$$\overline{\alpha} : \mathrm{Map}_{\mathcal{C}}(X, -) \longrightarrow F$$

through which all of the maps $\alpha : \mathrm{Map}_{\mathcal{C}}(x, -) \to F$ with $x \in \mathcal{C}^\kappa$ factor. Since $\mathcal{C}$ is κ-compactly-generated, we can write X as a κ-filtered colimit of κ-compact objects x_k; but F is also κ-accessible hence $\overline{\alpha} \in F(x_k)$ for some $k \in K$ meaning $\overline{\alpha}$ factors through $\mathrm{Map}(x_k, -)$.

But the restriction $G := F|_{\mathcal{C}^\kappa}$ is a colimit of corepresentable functors $\mathrm{Map}(y_k, -)$, $y_k \in \mathcal{C}^\kappa$ which provides a map $G \to \mathrm{Map}(X, -)$ and therefore a diagram

$$G \longrightarrow \mathrm{Map}_{\mathcal{C}}(X, -) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x_k, -) \longrightarrow G$$

We claim that this composite is the identity. It suffices to check that after precomposing by $\mathrm{Map}(y_k, x-) \to G$, we recover the canonical map to G; but this follows from the universal property of X: any composite $\mathrm{Map}(x_k, -) \to G$ factors uniquely through $\mathrm{Map}(X, -)$.

To conclude, note that $\mathcal{C}^\kappa$ is idempotent-complete: it is closed under retracts in a category with small colimits. In consequence, we find a natural equivalence $\eta : G \simeq \mathrm{Map}(x'_k, -)$ where $x'_k \in \mathcal{C}^\kappa$ is the retract of x_k corresponding to the previously built idempotent. The claim now follows since both the source and the target of η are κ-accessible and therefore left Kan extended from their restriction to the κ-compact objects of $\mathcal{C}$. □

Corollary 1.3.15 — Adjoint Functor Theorem. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable categories, then

- F has a right adjoint if and only if it preserves small colimits
- F has a left adjoint if and only if it preserves small limits and is accessible

Proof. We prove the second point. For any $X \in \mathcal{D}$, the composite $\mathrm{Map}(X, F(-)) : \mathcal{C} \to \mathcal{S}$ preserves small limits and is accessible hence is corepresented by some object suggestively denoted $F^L(X)$. But note that the natural equivalence

$$\mathrm{Map}(X, F(-)) \simeq \mathrm{Map}(F^L(X), -)$$

upgrades the $F^L(X)$ to a functor in X, since it shows the functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{P}(\mathcal{C}^{\mathrm{op}})$ lands pointwise in corepresentables (namely by $F^L(X)$), hence factor through some $\mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ which is our F^L up to a op. A dual argument deals with the other point. □

Definition 1.3.16 We let Pr^{L} denote the category of presentable categories and colimit preserving functors. We write Pr^{R} for the category with the same objects but limit-preserving accessible functors.

Since taking adjoint is a functorial process, we get an equivalence $\mathrm{Pr}^{\mathrm{L}} \simeq (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$.

Lemma 1.3.17 The categories Pr^{R} and Pr^{R} have small limits and they are preserved by the forgetful functors $\mathrm{Pr}^{\mathrm{R}} \to \mathbf{Cat}$, $\mathrm{Pr}^{\mathrm{L}} \to \mathbf{Cat}$. In consequence, they also have small colimits which are computed by taking adjoints everywhere and then forming the limit in $\mathbf{Cat}$.

Proof. By Proposition 1.3.3, we know that accessible categories are closed under limits along accessible functors in $\mathbf{Cat}$. Hence, to conclude, we have to show that given a diagram $\mathcal{C} : K \to \mathbf{Cat}$ in either category, $\lim_{k\in K} \mathcal{C}_k$ has small colimits and the projection functors

$$\pi_k : \lim_{k\in K} \mathcal{C}_k \longrightarrow \mathcal{C}_k$$

preserve small limits if the transition map do and small colimits if the transition maps do.

By Lemma 1.1.2 and its dual for limits, $\lim_{k\in K} \mathcal{C}_k$ is equivalently given by $\mathrm{Fun}_{/A}(A, \mathrm{Un}(\mathcal{C})) \simeq \mathrm{Fun}_{/A^{\mathrm{op}}}(A^{\mathrm{op}}, \mathrm{Un}^{cart}(\mathcal{C}))$; the colimit claim now follows from [Lur18a, Corollary 7.1.10.3/06AR] and the limit one is deduced from the other by a game of op's. □

In particular, we get from the above a higher categorical kind of additivity:

Corollary 1.3.18 Let K be a space and $F : K \to \mathrm{Pr}^{\mathrm{L}}$ be a functor. Then, there is a canonical equivalence

$$\mathrm{colim}_K F \xrightarrow{\simeq} \lim_K F$$

In particular, in Pr^{L}, arbitrary products and coproducts coincide.

We now turn to the symmetric monoidal structure of Pr^{L}. We first note that if $\mathcal{C}_0, \mathcal{D}_0$ are categories with κ-small colimits, then there is a category $\mathcal{C}_0 \otimes^\kappa \mathcal{D}_0$ with κ-small colimits and a map $\mathcal{C}_0 \times \mathcal{D}_0 \to \mathcal{C}_0 \otimes^\kappa \mathcal{D}_0$ satisfying the following universal property

$$\mathrm{Fun}^{\kappa-\mathrm{colim}}(\mathcal{C}_0 \otimes^\kappa \mathcal{D}_0, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\kappa-\mathrm{colim}}(\mathcal{C}_0, \mathrm{Fun}^{\kappa-\mathrm{colim}}(\mathcal{D}_0, \mathcal{E}))$$

for every $\mathcal{E}$ with κ-small colimits. This follows readily from Theorem 1.2.2. Of course, this holds more generally for all small colimits, and the following claims it restricts nicely to the presentable setting:

Proposition 1.3.19 Let $\mathcal{C}, \mathcal{D}$ be presentable categories, then there is a presentable category $\mathcal{C} \otimes \mathcal{D}$ and a functor $\eta : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ preserving colimits in each variable which induces for any presentable $\mathcal{E}$:

$$\eta^* : \mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}))$$

Moreover, $(\mathrm{Pr}^{\mathrm{L}}, \otimes)$ is a symmetric monoidal category with unit $\mathcal{S}$, which restricts to the full subcategories $\mathrm{Pr}^{\mathrm{L}}_\kappa$ spanned by κ-compactly-generated categories. In particular, if $\mathcal{C}, \mathcal{D}$ are compactly-generated, so is $\mathcal{C} \otimes \mathcal{D}$ and η sends $\mathcal{C}^\kappa \times \mathcal{D}^\kappa$ to κ-compact objects.

Proof. The existence of such a category follows readily from Theorem 1.2.2, by freely adding all small colimits to $\mathcal{C} \times \mathcal{D}$ while enforcing that colimits in one variable are colimits. In particular, we note that $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ is not fully-faithful but verifies the universal property:

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

where the superscript biL refers to the full subcategory of those functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits in each variable, which is equivalent to the wanted category.

By *fiat*, $\mathcal{C} \otimes \mathcal{D}$ has all small colimits. We check it is accessible: the explicit construction of Theorem 1.2.2 shows that $\mathcal{C} \times \mathcal{D}$ identifies with a full subcategory of presheaves on $\mathcal{C} \times \mathcal{D}$ spanned by a collection of local objects, namely those $\phi : \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$ which send colimits in either $\mathcal{C}$ or $\mathcal{D}$ to limits. Fixing a κ such that both $\mathcal{C}$ and $\mathcal{D}$ are κ-compactly generated, we find

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa \otimes^\kappa \mathcal{D}^\kappa)$$

from which the presentability is immediate, and in fact the restriction to the full subcategory of κ-compactly generated categories as well. Note in particular that the image of $\mathcal{C}^\kappa \times \mathcal{D}^\kappa$ lands in the κ-compact generators of $\mathcal{C} \otimes \mathcal{D}$.

The universal property of $\mathcal{S}$ from Proposition 1.1.5 implies readily the unit claim. More generally, the symmetric monoidal structure is induced by the cartesian one on $\widehat{\mathbf{Cat}}$, so for instance the symmetry comes from the equivalence $\mathcal{C} \times \mathcal{D} \simeq \mathcal{D} \times \mathcal{C}$. We leave the reader figure out how to make this into a precise proof. We note however that $\mathcal{S}$ is ω-compactly generated and the tensor product preserves κ-compactly generated categories hence this structure restricts. □

Remark 1.3.20 There is a more explicit formula for the tensor product $\mathcal{C} \otimes \mathcal{D}$, namely as the functor category $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$. Indeed, note that the expression of the above proof also identifies $\mathcal{C} \otimes \mathcal{D}$ as $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}))$. By Lemma 1.3.12, this category is equivalent to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.

Note that we had already seen that $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{D}$, even before the adjoint functor theorem.

The universal property of the tensor product implies that $\mathcal{P}(\mathcal{C}) \otimes \mathcal{P}(\mathcal{D}) \simeq \mathcal{P}(\mathcal{C} \times \mathcal{D})$, which upgrades to show that $\mathcal{P} : \mathbf{Cat} \to \mathrm{Pr}^{\mathrm{L}}$ is symmetric monoidal. More generally, Ind_κ acquires a symmetric monoidal structure from small categories with κ-small colimits and $\otimes^\kappa$ as tensor product to κ-compactly generated categories with the above restricted tensor product.

Lemma 1.3.21 Suppose $\mathcal{C}, \mathcal{D}$ are presentable, then the category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ of left adjoint functors is presentable. In particular, Pr^{L} is closed.

Proof. Note that $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ is cocomplete with pointwise colimits so we only have to focus on accessibility. Fix κ such that $\mathcal{C}, \mathcal{D}$ are κ-compactly generated, then we have

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\kappa-\mathrm{colim}}(\mathcal{C}^\kappa, \mathcal{D})$$

By Proposition 1.3.3, the category $\mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D})$ is accessible. Fix $p : K \to \mathcal{C}^\kappa$ a small diagram, then we claim that the full subcategory of $\mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D})$ which send p to a colimit diagram is accessible and so is the inclusion; indeed, there is a pullback square

$$\begin{array}{ccc} \mathrm{Fun}^{p\to\mathrm{colim}}(\mathcal{C}^\kappa, \mathcal{D}) & \longrightarrow & \mathrm{Fun}^{\mathrm{colim}}(K^\triangleright, \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(K^\triangleright, \mathcal{D}) \end{array}$$

where the top right category is spanned by those colimit diagrams $K^\triangleright \to \mathcal{D}$. In particular, this category is equivalent to $\mathrm{Fun}(K, \mathcal{D})$ and the right vertical map has a right adjoint so that the above pullback is of accessible categories, hence $\mathrm{Fun}^{p\to\mathrm{colim}}(\mathcal{C}^\kappa, \mathcal{D})$ is itself accessible by Proposition 1.3.3 and the map to $\mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D})$ accessible as well.

But now, the wanted category is again a limit, along accessible functors, over all the possible choices of p. Another instance of Proposition 1.3.3 concludes. □

In particular, Pr^{L} is enriched in itself. In turn, more is true: $\mathrm{Pr}^{\mathrm{L}}_\kappa$ *is* κ-compactly generated (in particular presentable); we refer to introductory lecture of [Sch25] for a deeper look at this.

1.4 Dualizable and compactly-generated stable categories

We write $\mathbf{Cat}^{\mathrm{Ex}}$ for the category of stable categories and exact functors between them and $\mathbf{Cat}^{\mathrm{perf}}$ for the full subcategory of idempotent-complete such categories; this category is reflexive and coreflexive. Thomason's cofinality theorem describes the fiber of $\mathrm{Idem} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{perf}}$, the idempotent-completion functor which is left adjoint to the inclusion: over $\mathcal{C}$, it is in bijection with subgroups of $K_0(\mathcal{C})$ ordered by inclusion.

Recall that given a pointed category with finite limits $\mathcal{C}$, the terminal stable category $\mathrm{Sp}(\mathcal{C})$ with a finite-limit preserving functor $\mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is given by the following limit:

$$\mathrm{Sp}(\mathcal{C}) \simeq \lim\left(\ldots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right)$$

and the functor is projection to the last term that we denote Ω^∞. Dually, if $\mathcal{C}$ is pointed has finite colimits, there is an initial category $\mathrm{SW}(\mathcal{C})$, sometimes called the *Spanier-Whitehead stabilization*, with a finite-colimit preserving functor $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ given by

$$\mathrm{SW}(\mathcal{C}) \simeq \mathrm{colim}\left(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \ldots \right)$$

and we write $\Sigma^\infty : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$ for the induced map from the first term; here the colimit has to be taken in the category of finitely cocomplete categories and finite-colimit preserving functors. In general, even if $\mathcal{C}$ has finite limits and colimits, these two categories do not agree. Indeed, note that an object of $\mathrm{SW}(\mathcal{C})$ is always a *finite* desuspension of an object of $\mathcal{C}$ whereas $\mathrm{Sp}(\mathcal{C})$ may have objects which are never desuspensions of an object of $\mathcal{C}$. This is already the case for $\mathcal{C} = \mathcal{S}$.

However, the following is always true:

Lemma 1.4.1 Let $\mathcal{C}$ be a category with finite colimits, then there is a canonical equivalence

$$\mathrm{Sp}(\mathrm{Ind}(\mathcal{C})) \simeq \mathrm{Ind}(\mathrm{SW}(\mathcal{C}))$$

In particular, the stabilization functor Sp restrict to a right adjoint $\mathrm{Pr}^{\mathrm{R}} \to \mathrm{Pr}^{\mathrm{R}}_{\mathrm{Ex}}$ to the inclusion hence a left adjoint $\mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ to the inclusion.

Proof. The functor $\mathrm{Ind} : \mathbf{Cat}^{\mathrm{fincolim}} \to \mathrm{Pr}^{\mathrm{L}}$ is left adjoint to the inclusion, so preserves colimits, hence

$$\mathrm{Ind}(\mathrm{SW}(\mathcal{C})) \simeq \operatorname{colim}\left(\mathrm{Ind}(\mathcal{C}) \xrightarrow{\Sigma} \mathrm{Ind}(\mathcal{C}) \xrightarrow{\Sigma} \mathrm{Ind}(\mathcal{C}) \xrightarrow{\Sigma} \dots \right)$$

but one has to be careful that this colimit is now taken in Pr^{L}, which are computed by taking right adjoints and forming the limit by Lemma 1.3.17. But doing this, we recover precisely the formula for $\mathrm{Sp}(\mathrm{Ind}(\mathcal{C}))$ which concludes. □

Let us note that in particular, the canonical functor $\Omega^\infty : \mathrm{Sp}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})$ is necessarily right adjoint to $\mathrm{Ind}(\Sigma^\infty) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{SW}(\mathcal{C}))$. In particular, Ω^∞ is accessible. If $\mathcal{C}$ is stable, then so is $\mathrm{Ind}(\mathcal{C})$ hence defines a functor $\mathrm{Ind} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$. This functor factors through the idempotent-completion one Idem and the resulting $\mathrm{Ind} : \mathbf{Cat}^{\mathrm{perf}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ is faithful with image those categories which are compactly-generated and functors preserve compact objects.

■ **Example 1.4.2** The category of spectra $\mathrm{Sp} := \mathrm{Sp}(\mathcal{S})$ is compactly-generated by $\mathrm{SW}(\mathcal{S}^{\mathrm{fin}})$, where $\mathcal{S}^{\mathrm{fin}}$ denotes the category of finite spaces. In particular, it follows that Sp is generated under small colimits by $\mathbb{S} := \Sigma^\infty(*)$, freely as a stable category, i.e.

$$\mathrm{ev}_{\mathbb{S}} : \mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(*, \mathcal{C})$$

for every cocomplete $\mathcal{C}$. ■

Remark 1.4.3 In fact, Lemma 1.4.1 is part of a more general phenomenon: if $\mathcal{C}$ is presentable, then so is $\mathrm{Sp}(\mathcal{C})$, because limits in Pr^{R} are computed underlying. In particular, Sp is a right adjoint to the inclusion $\mathrm{Pr}^{\mathrm{R}}_{\mathrm{Ex}} \subseteq \mathrm{Pr}^{\mathrm{R}}$, because it has the correct universal property already. Here, we used that a functor $\mathcal{D} \to \lim_K \mathcal{C}$ is accessible (resp. preserves limits) if and only if all the projections $\mathcal{D} \to \mathcal{C}$ do, as we have seen in Lemma 1.3.17.

The following result is often known under the name Schwede–Shipley. They proved in [SS03] a version of this result in model-category-land[2]; the ∞-categorical claim can be found in [Lur17, Theorem 7.1.2.1].

Theorem 1.4.4 — Schwede–Shipley. If $R \in \mathrm{Alg}(\mathrm{Sp})$, the category $\mathrm{Mod}(R)$ of (left) R-modules is compactly-generated and the object R viewed as a R-module jointly detects equivalences.

Moreover, any cocomplete stable category with a single compact generator X is equivalent to $\mathrm{Mod}(\mathrm{end}_{\mathcal{C}}(X))$.

We will not prove this right now. We will prove a more precise version of the above in Proposition **??**, but this requires some enriched category theory, hence it will have to wait until we have the time to develop this material. We note already one main consequence of this more precise version, which can also be found in Proposition 7.1.2.4 of [Lur17] and is a generalization of a result of Eilenberg–Watts: any functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is given by tensoring by a (R, S)-bimodule.

In fact, one can recover Eilenberg–Watts from our more precise version of Schwede–Shipley (though maybe not from the above version) once it is combined with the fact proven in [BCKW25, Corollary 7.4.12] that for a discrete ring R, the map $\mathrm{Mod}(R) \to D(R)$ is the initial map with target a presentable stable category which preserves filtered colimits and exact sequences.

[2] This is like Munchkinland but the yellow brick road has been cofibrantly replaced.

Note that if $\{X_1, ..., X_n\} \subset \mathcal{C}^\omega$ jointly detect equivalences in a presentable stable $\mathcal{C}$, then $X_1 \oplus ... \oplus X_n$ is a single compact generator of $\mathcal{C}$. Hence, the presentable stable categories which are not of the form $\mathrm{Mod}(R)$ are somewhat wild: they do not have finitely many compact generators. In particular, since every category is the filtered colimit of its full subcategory generated by finitely many objects, we get:

Corollary 1.4.5 A compact object in $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{Ex}}$, the full subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ spanned by compactly-generated and compact-preserving maps, is necessarily of the form $\mathrm{Mod}(R)$.

Proof. A compact category $\mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ is necessarily a retract of a category generated by finitely many objects, hence itself finitely generated. If $\mathcal{C}$ is further compact in $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{Ex}}$, then running the colimit over compactly-generated subcategories abuts to the wanted conclusion, using that the retraction preserves compact objects by *fiat*. □

Remark 1.4.6 The above discussion fails spectacularly without the stable hypothesis, as exemplified by Corollary 1.3.10.

Let us now investigate the symmetric monoidal structure:

Lemma 1.4.7 The canonical map $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ exhibits the category of spectra Sp as an idempotent object in $(\mathrm{Pr}^{\mathrm{L}}, \otimes)$, so in particular, Sp has a unique commutative algebra structure whose unit map is the previous one. In fact, a presentable category $\mathcal{C}$ is stable if and only if $\mathcal{C} \xrightarrow{\simeq} \mathrm{Sp} \otimes \mathcal{C}$.

Proof. The first statement follows from the second, using that Sp is stable. Recall from Remark 1.3.20 that

$$\mathrm{Sp} \otimes \mathcal{C} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

In particular, this is a stable category so we get one side of the equivalence. On the other hand, if $\mathcal{C}$ is stable, then

$$\mathcal{C} \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

where the first equivalence is by Remark 1.3.20 and the second by Remark 1.4.3. □

It follows that $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ inherits a tensor product from Pr^{L} whose unit is Sp, the category of spectra. Moreover, the stabilization functor $\mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ left adjoint to the inclusion coincides with $\mathrm{Sp} \otimes -$. Note that we could have gone a different route to build this tensor product more concretely, by adapting the proof of Proposition 1.3.19.

In particular, adapting this technique shows that $\mathbf{Cat}^{\mathrm{Ex}}$ and $\mathbf{Cat}^{\mathrm{perf}}$ are symmetric monoidal categories with variants of the Lurie tensor product, and the same unit $\mathrm{Sp}^{\mathrm{fin}} \simeq \mathrm{Sp}^\omega$, the category of spectra obtained from finite colimits from $\mathbb{S}$, which happens to be idempotent-complete (this was proven in Fabian's lecture).

Lemma 1.4.8 The functor $\mathrm{Ind} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ is monoidal, with the monoidal structure canonically induced by its pointwise universal property.

Proof. Since we have been rather cavalier with building the monoidal structure, let also be cavalier with this claim: first, note that Ind commutes with products since these are computed as direct sums in both $\mathbf{Cat}^{\mathrm{Ex}}$ and $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$. Moreover, if $\mathcal{C}, \mathcal{D}$ are small stable categories and $\mathcal{E}$ is presentable then

$$\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{C} \otimes \mathcal{D}), \mathcal{E})$$

but at the same times, the left hand side category is equivalent to

$$\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}, \mathcal{E})) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{C}), \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{D}), \mathcal{E}))$$

so that $\mathrm{Ind}(\mathcal{C} \otimes \mathcal{D})$ and $\mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{D})$ have the same universal property under the product. Refining the argument lifts it to a monoidal structure. □

In particular, we find that if $\mathcal{C}, \mathcal{D}$ are idempotent-complete, $\mathcal{C} \otimes \mathcal{D} \simeq (\mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{D}))^\omega$, where the right hand side is the tensor of presentable categories. Note that even if $\mathcal{C}$ and $\mathcal{D}$ are idempotent-complete, their tensor product need not be idempotent-complete either: in general, the tensor in $\mathbf{Cat}^{\mathrm{Ex}}$ in the smallest stable subcategory of $\mathrm{Ind}(\mathcal{C} \otimes \mathcal{D})$ which contains the image of $\mathcal{C} \times \mathcal{D}$. Already if $\mathcal{C} = \mathrm{Sp}^{\mathrm{fin}}$ is stable and $\mathcal{D}$ any non-idempotent-complete category, these differ.

Definition 1.4.9 Let $\mathcal{C}^{\otimes} := (\mathcal{C}, \otimes)$ be a symmetric monoidal category. We say that X is *dualizable* if there exists an object $X^{\vee} \in \mathcal{C}$, two maps $\mathrm{ev} : X \otimes X^{\vee} \to \mathbb{1}$ and $\mathrm{coev} : \mathbb{1} \to X^{\vee} \otimes X$ and homotopies between the composites

$$X \xrightarrow{X \otimes \mathrm{coev}} X \otimes X^{\vee} \otimes X \xrightarrow{\mathrm{ev} \otimes X} X$$

and id_X as well as one between

$$X^{\vee} \xrightarrow{\mathrm{coev} \otimes X^{\vee}} X^{\vee} \otimes X \otimes X^{\vee} \xrightarrow{X^{\vee} \otimes \mathrm{ev}} X^{\vee}$$

and $\mathrm{id}_{X^{\vee}}$.

This may look like a structure on X, but it is actually a property. Indeed, consider the 2-category $B\mathcal{C}^{\otimes}$ with one point $*$ and $\mathcal{C}^{\otimes}$ with endomorphism category. Then, $X \in \mathcal{C}$ has an adjoint if and only if it is dualizable; the unit and counit are the evaluation and the coevaluation and the homotopies are the triangles identities. We can also do away with the 2-category and prove this directly, as in the following:

Lemma 1.4.10 Suppose that $\mathcal{C}^{\otimes}$ is a closed symmetric monoidal category. Then, if X is dualizable in $\mathcal{C}$, we have an equivalence

$$X^{\vee} \simeq \underline{\mathrm{Map}}(X, \mathbb{1})$$

where $\underline{\mathrm{Map}}$ denotes the internal mapping object. Moreover, an object X is dualizable if and only if the map

$$\underline{\mathrm{Map}}(X, \mathbb{1}) \otimes Y \longrightarrow \underline{\mathrm{Map}}(X, Y)$$

obtained by adjunction from $\mathrm{ev} \otimes Y : (\underline{\mathrm{Map}}(X, \mathbb{1}) \otimes X) \otimes Y \longrightarrow Y$, is an equivalence for every Y.

Proof. For the first claim, it suffices to check that if X is dualizable, then $- \otimes X$ is left adjoint to $- \otimes X^{\vee}$. We claim that the natural map as follows:

$$\mathrm{Map}(Y, Z \otimes X^{\vee}) \xrightarrow{- \otimes X} \mathrm{Map}(Y \otimes X, Z \otimes X^{\vee} \otimes X) \xrightarrow{(Z \otimes \mathrm{ev}_X)_*} \mathrm{Map}(Y \otimes X, Z)$$

is an equivalence. Its inverse is given by

$$\mathrm{Map}(Y \otimes X, Z) \xrightarrow{- \otimes X^{\vee}} \mathrm{Map}(Y \otimes X \otimes X^{\vee}, Z \otimes X^{\vee}) \xrightarrow{(Y \otimes \mathrm{coev}_X)^*} \mathrm{Map}(Y, Z \otimes X^{\vee})$$

and the homotopy witnessing that both composite are the identity are deduced from the triangle identities.

In fact, the above also works in the other direction: if $- \otimes X$ has a right adjoint of the form[3] $- \otimes X^{\vee}$, then X is dualizable. In particular, note that it follows that if X is dualizable, there is a chain of equivalences:

$$\mathrm{Map}(Y, \underline{\mathrm{Map}}(X, Z)) \simeq \mathrm{Map}(Y \otimes X, Z) \simeq \mathrm{Map}(Y, Z \otimes \underline{\mathrm{Map}}(X, \mathbb{1}))$$

induced by the wanted map. Reciprocally, since the first equivalence always holds, this also implies that $- \otimes \underline{\mathrm{Map}}(X, \mathbb{1})$ is right adjoint to $- \otimes X$ hence concludes. □

Note also that the proof of the first claim did not need the closure for the adjunction to hold. We are therefore also able to get very generally:

Corollary 1.4.11 Any $\otimes$-invertible object is dualizable, with dual its $\otimes$-inverse.

Since equivalences are closed under retracts, so are dualizable objects. Moreover, note that if $\mathcal{C}^{\otimes}$ is stable, then both sides of the equivalences are exact in X; resuming those properties, we get:

[3]There are counterexample to the statement when the right adjoint of $- \otimes X$ is not assumed to be of this form

Corollary 1.4.12 Dualizable objects are closed under retracts. If $\mathcal{C}^\otimes$ is stable, then they are also form a stable subcategory of $\mathcal{C}$

As the last bit of abstract definition about symmetric-monoidal categories, let us introduce the following definition. We say that a symmetric monoidal category $\mathcal{C}$ is *presentably symmetric monoidal* if $\mathcal{C}$ is presentable and the tensor product commutes with colimits in both variables, i.e. if $\mathcal{C}^\otimes$ lifts to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}})$.

Definition 1.4.13 A stable, compactly-generated category $\mathcal{C}$ with a presentably symmetric monoidal structure is said to be *rigid* if every compact object is dualizable and $\mathbb{1}_\mathcal{C}$ is compact.

Presentably symmetric monoidal categories are always closed: the functor $X \otimes -$ preserves small colimits hence by Corollary 1.3.15, it has a right adjoint.

Readers aware of a different definition, tailored to smaller categories, will be happy to hear about the following lemma:

Lemma 1.4.14 Suppose $\mathcal{C}^\otimes$ is small such that every object is dualizable. Then, $\mathrm{Ind}(\mathcal{C})^\otimes$ is rigid.

Proof. We note that the monoidality claim of Lemma 1.4.8 adapts straightforwardly to non-stable categories when replacing the source with $\mathbf{Cat}^{\mathrm{REx}}$ of finitely-cocomplete categories and right-exact functors between them. Therefore, if $\mathcal{C}^\otimes \in \mathrm{CAlg}(\mathbf{Cat}^{\mathrm{REx}})$, then $\mathrm{Ind}(\mathcal{C})^\otimes := \mathrm{Ind}(\mathcal{C}^\otimes) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ in such a way that the symmetric monoidal structure restricts to the one on $\mathcal{C}^\otimes$ via Yoneda embedding (in particular, the units coincide). It follows that the image of $\mathcal{C}$ is dualizable, and it suffice to remark that dualizable objects are necessarily closed under retracts to conclude. □

■ **Example 1.4.15** The category $\mathcal{S}$ is not rigid, since $\mathrm{Map}(X, *) \simeq *$ for every X. In fact, no cartesian closed category is, unless they are trivial.

On the other hand, when endowed with its canonical structure, the category Sp is rigid: indeed, recall that $\mathrm{Sp}^\omega \simeq \mathrm{Sp}^{\mathrm{fin}}$ and that X is dualizable if and only if

$$\mathrm{map}(X, \mathbb{S}) \otimes Y \simeq \mathrm{map}(X, Y)$$

is an equivalence for all Y. But both functors $\mathrm{map}(X, \mathbb{S}) \otimes -$ and $\mathrm{map}(X, -)$ preserve colimits when X is compact, because Sp is stable and the above holds for $Y = \mathbb{S}$, hence generally. In fact, note that if X is dualizable, then $\mathrm{map}(X, \mathbb{S}) \otimes -$ preserves all small colimits, therefore since $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ preserves filtered colimits, X is compact. ■

Remark 1.4.16 The converse part of Example 1.4.15 holds more generally. Suppose $\mathbb{1}$ is compact and let X be dualizable in a presentably symmetric monoidal $\mathcal{C}$, then Lemma 1.4.10 provides an equivalence

$$\mathrm{Map}(X, -) \simeq \mathrm{Map}(\mathbb{1}, - \otimes \underline{\mathrm{Map}}(X, \mathbb{1}))$$

so that X is in fact compact.

Proposition 1.4.17 Let R be a ring spectrum, then $\mathrm{Mod}(R)$ is rigid.

Proof. Since $R \in \mathrm{Mod}(R)$ is compact, every dualizable is compact as well. Reciprocally, given a compact $M \in \mathrm{Mod}(R)$, then M induces a colimit-preserving functor $M \otimes - : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ which further preserves compact objects. In particular, its right adjoint given by Corollary 1.3.15 preserves filtered colimits, but it is also exact. Hence, using Schwede–Shipley (but not Theorem 1.4.4, the stronger Proposition **??** which implies the higher categorical Eilenberg–Watts), it is of the form $N \otimes -$. This is precisely the dualizability criterion of Lemma 1.4.10, which concludes. □

■ **Example 1.4.18** The category Sp^{BS^1} is not rigid: the unit is the constant functor equal to $\mathbb{S}$, i.e. the sphere with the trivial S^1-action which is *not* compact. Differently stated, its right adjoint $(-)^{\mathrm{h}S^1}$ does not preserve filtered colimits.

To see this, note that every S^1-spectrum is a filtered colimit of finite colimits of induced S^1-objects, i.e. of the form $S^1 \otimes X$ with the action on the left factor; this is because taking induced objects is left adjoint to a conservative functor. But $\mathbb{S}^{\text{triv}}$ not a retract of such finite colimits of induced objects, since such a retraction cannot be compatible with the actions. ■

Our goal in the rest of this section is study dualizable objects of $\mathbf{Cat}^{\text{Ex}}, \mathbf{Cat}^{\text{perf}}$ and $\text{Pr}^{\text{L}}_{\text{Ex}}$ (or more precisely, to barely scratch the surface in their study). Before that however, we make a small detour to collect some facts about coends.

If $\mathcal{C}$ is a category, we write $\text{TwAr}(\mathcal{C})$ for the twisted arrow category on $\mathcal{C}$: our convention is that $\text{TwAr}(\mathcal{C}) \to \mathcal{C}^{\text{op}} \times \mathcal{C}$ is the right fibration classifying $\text{Map}_{\mathcal{C}}$ and therefore, a map from $X \to X'$ to $Y \to Y'$ in $\text{TwAr}(\mathcal{C})$ is a diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \uparrow \\ Y & \longrightarrow & Y' \end{array}$$

exhibiting the source as factoring through the target. This convention is one of the two possible (the other taking the *left* fibration); we made this choice so that the following definition is nicer for the objects we will use the most; the dual convention switches $\text{TwAr}(\mathcal{C})$ and $\text{TwAr}(\mathcal{C})^{\text{op}}$.

Definition 1.4.19 Let $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor, then if it exists, the *coend* of F is the object of $\mathcal{D}$ defined as follows:

$$\int^{X \in \mathcal{C}} F(X, X) := \underset{(p:X \to Y) \in \text{TwAr}(\mathcal{C})}{\text{colim}} F(Y, X)$$

The *end* is $\sout{\text{near}}$ the following object

$$\int_{X \in \mathcal{C}} F(X, X) := \lim_{(p:X \to Y) \in \text{TwAr}(\mathcal{C})^{\text{op}}} F(X, Y)$$

One of the most important example of ends is the space of natural transformations. Note that for every $X \in \mathcal{C}$, there is a map $\text{Nat}(F, G) \to \text{Map}_{\mathcal{D}}(F(X), G(X))$ evaluating the natural transformation. This collection of maps upgrades to a map

$$\text{Nat}(F, G) \longrightarrow \int_{X \in \mathcal{C}} \text{Map}_{\mathcal{D}}(F(X), G(X))$$

Indeed, given a map $p : X \to Y$, we can refine the above map to $\text{Nat}(F, G) \to \text{Map}_{\mathcal{D}}(F(X), G(Y))$ by sending η to $F(X) \to G(X) \to G(Y)$, and we leave the reader write out the proper way to make this functorial in maps of $\text{TwAr}(\mathcal{C})^{\text{op}}$. We claim:

Lemma 1.4.20 Given $F, G : \mathcal{C} \to \mathcal{D}$, the above map is an equivalence, i.e.

$$\text{Nat}(F, G) \simeq \int_{X \in \mathcal{C}} \text{Map}_{\mathcal{D}}(F(X), G(X))$$

Proof. The proof strategy is that of [GHN17]. By the dual of Lemma 1.1.2, the limit of the functor

$$\text{TwAr}(\mathcal{C})^{\text{op}} \longrightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}^{\text{op}} \xrightarrow{(F,G)} \mathcal{D}^{\text{op}} \times \mathcal{D} \xrightarrow{\text{Map}} \mathcal{S}$$

is the category of sections of the cocartesian fibration (which is just a left fibration). Using that such fibrations are closed under pullbacks, we get that this category is equivalent to the category of lifts

$$\begin{array}{ccccc} & & & & \text{TwAr}(\mathcal{D}) \\ & & & \nearrow & \downarrow \\ \text{TwAr}(\mathcal{C}) & \longrightarrow & \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{(F,G)} & \mathcal{D}^{\text{op}} \times \mathcal{D} \end{array}$$

This is equivalently a map between left fibrations, by pulling back $\text{TwAr}(\mathcal{D})$ over $\mathcal{C}^{\text{op}} \times \mathcal{C}$; in particular, by straightening, we have gotten an equivalence

$$\text{Nat}(\text{Map}_{\mathcal{C}}, \text{Map}_{\mathcal{D}}(F, G)) \simeq \int_{X \in \mathcal{C}} \text{Map}_{\mathcal{D}}(F(X), G(X))$$

Currying in the contravariant variable, we see that the left hand side is also

$$\mathrm{Nat}(j_{\mathcal{C}}, F^* \circ j_{\mathcal{D}} \circ G)$$

in the category of functors $\mathcal{C} \to \mathcal{P}(\mathcal{C})$. We conclude by remarking that left Kan extension along F of a presheaf represented by X is represented by $F(X)$, so that using the adjunction between left Kan extension $F_!$ and F^* and the fully-faithfulness of the Yoneda embedding, we deduce

$$\mathrm{Nat}(F, G) \simeq \int_{X \in \mathcal{C}} \mathrm{Map}_{\mathcal{D}}(F(X), G(X))$$

We let the reader track the canonical maps from the right hand side to each $\mathrm{Map}(F(X), G(Y))$ along (un)straighening to see that we proven that the correct map realizes the above. ☐

Remark 1.4.21 Suppose $\mathcal{D}$ is stable, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is stable and the spectrum of natural transformations satisfies

$$\mathrm{nat}(F, G) \simeq \int_{X \in \mathcal{C}} \mathrm{map}_{\mathcal{D}}(F(X), G(X))$$

Indeed, the right hand side is exact in both F and G and since Ω^∞ commutes with ends, recovers the space of natural transformation by the above.

Lemma 1.4.22 — Coend Yoneda Lemma. Any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ satisfies:

$$F(-) \simeq \int^{X \in \mathcal{C}} F(X) \times \mathrm{Map}_{\mathcal{C}}(-, X)$$

If $\mathcal{C}$ is stable, then the above result also applies to exact $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ as follows:

$$F(-) \simeq \int^{X \in \mathcal{C}} F(X) \otimes \mathrm{map}_{\mathcal{C}}(-, X)$$

where the tensor product is pointwise in Sp and $\mathrm{map}_{\mathcal{C}}$ the spectrally enriched mapping object.

Proof. We remark that

$$\mathrm{Map}\Big(\int^{X \in \mathcal{C}} F(X) \times \mathrm{Map}_{\mathcal{C}}(Y, X), Z\Big) \simeq \int_{X \in \mathcal{C}} \mathrm{Map}(F(X) \times \mathrm{Map}_{\mathcal{C}}(Y, X), Z)$$

Using the expression of the space of natural transformations we gave previously, the latter term is equivalently given by

$$\mathrm{Nat}(\mathrm{Map}_{\mathcal{C}}(Y, -), \mathrm{Map}(F(-), Z))$$

The actual Yoneda lemma now implies that this is none other than $\mathrm{Map}(F(Y), Z)$ and this chain of equivalence is natural in Y, Z, which concludes. The stable case is in all point similar, using the spectral Yoneda, the tensor product in Sp and Remark 1.4.21. ☐

Finally, we reach the content we wanted to explain in this section:

Proposition 1.4.23 Let $\mathcal{C}$ be a small stable category, then $\mathrm{Ind}(\mathcal{C})$ is dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ with dual $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$.

Proof. Using the monoidality of Ind of Lemma 1.4.8, we are reduced to specify two colimit-preserving functors $\mathrm{ev} : \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \to \mathrm{Sp}$ and $\mathrm{coev} : \mathrm{Sp} \to \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}})$ and prove the relevant identities hold. The second map is just a point in $\mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$ and we take the one corresponding to $\mathrm{map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$, the mapping spectra functor. For the first map, we take the unique extension of $\mathrm{map}_{\mathcal{C}}$ to a colimit-preserving functor out of $\mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}})$ instead.

Let us now give a more concrete description of those two functors. We claim that the functor $\mathrm{ev} : \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \to \mathrm{Sp}$ takes an exact presheaf $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$ and computes its coend:

$$\int^{X \in \mathcal{C}} F(X, X) \simeq \underset{(p : X \to Y) \in \mathrm{TwAr}(\mathcal{C})}{\mathrm{colim}} F(Y, X)$$

If F is represented by (X, Y), then it coincides with

$$\mathrm{map}_{\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}}(-,(X,Y)) \simeq \mathrm{map}_{\mathcal{C}}(X,-)\otimes \mathrm{map}_{\mathcal{C}}(-,Y)$$

thus this coend coincides with $\mathrm{map}(X, Y)$ by the coend Yoneda lemma 1.4.22. Moreover, the functor $F \mapsto \int^{X\in\mathcal{C}} F(X,X)$ is colimit-preserving, hence it coincides globally with the ev we picked.

Finally, we check that the following composite is the identity (and leave the other one to the reader)

$$\mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C})$$

which sends $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ to the presheaf

$$X \longmapsto \int^{Y\in\mathcal{C}} \mathrm{map}_{\mathcal{C}}(X,Y)\otimes F(Y)$$

The coend Yoneda lemma 1.4.22 gives precisely the equivalence we are looking for. □

One could also envision a strategy relying on Lemma 1.4.10 to prove Proposition 1.4.23. We will need the more explicit formula of the evaluation and the coevaluation which is why we wrote the proof this way; but the other strategy is generally useful to see what fails for higher cardinals, as we explain in the following Remark.

Remark 1.4.24 The analogue of Proposition 1.4.23 generally fails for Ind_κ when $\kappa > \omega$. The clearest claim is the identification of the dual: $\mathrm{Ind}_\kappa(\mathcal{C}^{\mathrm{op}})$ does not coincide with $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathrm{Sp})$ if $\kappa > \omega$, because one is κ-small colimits preserving functor $\mathcal{C} \to \mathrm{Sp}$ and the other is κ-small limit preserving functors $\mathcal{C} \to \mathrm{Sp}$.

In fact, by Lemma 1.4.10, we can check whether $\mathrm{Ind}_\kappa(\mathcal{C})$ is dualizable by comparing

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathrm{Sp})\otimes\mathcal{D} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathrm{Sp}))$$

and

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp}))$$

But of course, we cannot curry because neither colimits in right-adjoint functors nor limits in left-adjoint functors are computed pointwise. When $\kappa = \omega$, the miracle of stable categories happens: finite limits and finite colimits commute and the commutation is legitimate (hence realizing the aforementioned proof).

Let us also mention that there are more dualizable objects in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ than $\mathrm{Ind}(\mathcal{C})$ for a small $\mathcal{C}$: every kernel of a strongly-continuous localization between compactly-generated category is dualizable, as a retract in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ of a compactly-generated hence dualizable category. However, not all of them must be compactly-generated; examples abound and many predate the advent of higher category theory (almost ring theory, sheaves of spectra on a locally compact Hausdorff space, many kernels of localisations of derived categories of rings), we refer for instance to [Efi24, Section 1.5] for more material.

Finally, we dedicate the rest of this section to answering dualizability questions in categories closely related to $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$: the first comes at the insistence of a secret admirer[4] of this course.

In [Ste20, Proposition 5.1.4], Stefanich shows that a category $\mathcal{C}$ in $\mathbf{CAT}^{\mathrm{colim}}$, the (very large) category of large categories with small colimits and small colimits preserving functors, is presentable if and only if it is k_0-compact for k_0 the smallest large cardinal. The tensor product of Pr^{L} is compatible with the one of $\mathbf{CAT}^{\mathrm{colim}}$ (essentially by construction given the proof of Proposition 1.3.19), hence since $\mathcal{S}$ is itself presentable, Lemma 1.4.10 guarantees that all the dualizable objects of $\mathbf{CAT}^{\mathrm{colim}}$ are presentable. The stable analogue of the above follows immediately from a version of Stefanich's result, whose proof can be done in the exact same way.

Having dealt with the very large, let us also discuss other, smaller analogues of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$. We introduce the following definitions, originally by Kontsevich:

[4] Only his admiration is secret

Definition 1.4.25 A category $\mathcal{C}$ is said to be *proper* if $\mathrm{map}_{\mathcal{C}}$ lands in $\mathrm{Sp}^{\mathrm{fin}} = \mathrm{Sp}^{\omega}$.

A category $\mathcal{C}$ is said to be *smooth*, resp. *finite-smooth*, if $\mathrm{map}_{\mathcal{C}}$ is a compact object of $\mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}})$, resp. in the image of the Yoneda embedding of $\mathcal{C} \otimes \mathrm{Idem}(\mathcal{C}^{\mathrm{op}})$.

Since dualizability is a property and $\mathrm{Ind}(-)$ preserves dualizable objects, dualizability in $\mathbf{Cat}^{\mathrm{Ex}}$ and $\mathbf{Cat}^{\mathrm{perf}}$ is only a question about whether the objects and functors we defined in Proposition 1.4.23 descend to either finite or compact objects.

An idempotent-complete category is fully determined by its Ind-completion via taking compact objects; furthermore every dual in $\mathbf{Cat}^{\mathrm{Ex}}$ is idempotent-complete because it must be of the form $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Sp}^{\mathrm{fin}})$ where $\mathrm{Sp}^{\mathrm{fin}}$ is idempotent-complete. This has the surprising consequence that every dualizable category in $\mathbf{Cat}^{\mathrm{Ex}}$ is automatically idempotent-complete. Hence we have:

Corollary 1.4.26 Dualizable objects of $\mathbf{Cat}^{\mathrm{Ex}}$ are precisely the finite-smooth and proper categories. Dualizable objects of $\mathbf{Cat}^{\mathrm{perf}}$ are precisely the smooth and proper categories. In both cases, the dual category is the opposite and every such category is automatically idempotent-complete.

Remark 1.4.27 In particular, because they must be idempotent-complete, finite-smooth categories have the property that $\mathrm{map}_{\mathcal{C}}$ lands in the full subcategory of $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C})$ spanned by $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}$; because this is a stronger property, we could have alternatively defined finite-smooth this way and at least under the properness assumption, gotten the same property.

Since the unit $\mathrm{Sp}^{\mathrm{fin}}$ of both categories is compact, (finite-)smooth and proper implies compact, so in particular by the same argument as Corollary 1.4.5, they are either of the form $\mathrm{Mod}(R)^{\mathrm{fin}}$, the smallest stable subcategory of $\mathrm{Mod}(R)$ containing R, in the first case or $\mathrm{Perf}(R)$, its idempotent-completion, in the second.

In the other direction, compact objects of $\mathbf{Cat}^{\mathrm{perf}}$ are smooth by an argument due to Toën and Vezzosi (see also [KNP24, Lemma 4.7.4]); note that smoothness only depends on the idempotent-completion and $\mathrm{Idem}(-)$ preserves compact objects hence this also holds for compact objects of $\mathbf{Cat}^{\mathrm{Ex}}$. Compact on the other hand does not imply proper: let $\mathrm{Fun}(\mathrm{B}S^2, \mathrm{Sp})^{\omega}$ be the category of spectra with a S^2-action. Then, since S^2 is compact, so is this category and it is actually generated under finite colimits and retracts by $\mathbb{S}[\Omega S^2]$. But the underlying spectrum of $\mathbb{S}[\Omega S^2]$ is not bounded-below hence cannot be compact (as any finite colimit or retract of $\mathbb{S}$ is bounded-below) — in particular, $\mathrm{Fun}(\mathrm{B}S^2, \mathrm{Sp}^{\mathrm{fin}})$ is not proper.

■ **Example 1.4.28** We had seen in Example 1.4.15 that $\mathrm{Sp}^{\mathrm{fin}}$ was finite-smooth (it is even a pure tensor) and it is clearly proper since $\mathrm{map}(\mathbb{S}, \mathbb{S}) = \mathbb{S} \in \mathrm{Sp}^{\mathrm{fin}}$ and this therefore holds for every finite colimit of $\mathbb{S}$ in each variable. We already knew that $\mathrm{Sp}^{\mathrm{fin}}$ was dualizable in $\mathbf{Cat}^{\mathrm{Ex}}$ since it is the unit, so everything is coherent so far. ■

■ **Example 1.4.29** Let X be a smooth and proper scheme. Then, $\mathrm{Perf}(X)$ is a smooth and proper category and the converse holds by work of Kontsevich[5]. We refer to Lurie's [Lur18b, Chapter 11], for instance Proposition 11.3.2.4 or Theorem 11.4.0.3, and more generally the whole chapter for a discussion of smooth and properness.

The notion of finite-smoothness is significantly more restrictive: it contains the case where X admits a full exceptional collection, or differently stated where $\mathrm{Perf}(X)$ is the smallest subcategory of $\mathbf{Cat}^{\mathrm{Ex}}$ containing $\mathrm{Sp}^{\mathrm{fin}}$ and $\mathrm{Ar}(\mathrm{Sp}^{\mathrm{fin}})$ and closed under semi-orthogonal decompositions. We do not know whether this is necessary. ■

Let us reiterate that dualizable objects of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ are not necessarily compactly-generated categories; their first thorough study was done in [Lur18b]. In recent years, a lot of development has been happening in this direction, impulsed by ideas of Efimov, see for instance [Efi24]. It is not our intent to give a course on such ideas, and we refer instead to the course notes by Krause–Nikolaus–Pützstück [KNP24] instead.

[5] The only reference I am aware of by Kontsevich himself is a talk he gave for the birthday conference of Deligne in 2005, a video of which is available (at the time of writing) on the IAS website, and notes can be found on the IHES website.

Remark 1.4.30 Observe that a smooth and proper category has the following property: a copresheaf $\mathcal{C} \to \mathrm{Sp}$ is corepresentable if and only if it lands in $\mathrm{Sp}^{\mathrm{fin}}$. This is because the identification of the dual $\mathcal{C}^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Sp}^{\mathrm{fin}})$; of course the dual statement applies to presheaves and representable functors since $\mathcal{C}^{\mathrm{op}}$ is again dualizable.

Essentially copying our proof of Corollary 1.3.15, we deduce that every exact functor $\mathcal{C} \to \mathcal{D}$ between smooth and proper categories has a left and a right adjoint (and therefore infinitely many in each direction) — the first fact actually only requires $\mathcal{D}$ to be proper. Therefore, every object of a smooth and proper category is compact since $\mathrm{map}(X, -)$ has a right adjoint (in particular, every filtered colimit in $\mathcal{C}$ must retracts onto one of its terms so there cannot be a lot of them).

In fact, the category of exact functors $\mathcal{C} \to \mathcal{D}$ between smooth and proper categories identifies with $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{D}$ via the functor induced from

$$(X, Y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longmapsto \mathrm{map}_{\mathcal{C}}(X, -) \otimes Y \in \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})$$

To understand the procedure of taking adjoints in the above, just note that this is a composite $\mathcal{C} \to \mathrm{Sp}^{\mathrm{fin}} \to \mathcal{D}$ where $\mathrm{Sp}^{\mathrm{fin}}$ is smooth and proper hence we are reduced to understand what is the right adjoint of $\mathrm{map}(X, -)$ or the left adjoint of $Y \otimes -$. This is in general non-trivial.

It is an interesting exercise, that we recommend to the reader to try to compute iteratively the left and the right adjoints of the source projection $s : \mathrm{Ar}(\mathrm{Sp}^{\mathrm{fin}}) \to \mathrm{Sp}^{\mathrm{fin}}$, which is the functor corepresented by $\mathrm{id}_{\mathbb{S}} : \mathbb{S} \to \mathbb{S}$. This gives a non-periodic infinite chain of adjunction, but with period six, the next adjoint is given by a shift (either a suspension for left adjoints or a loop for right adjoint). Interpreting an arrow f as a diagram

$$\dots \longrightarrow \Omega Y \longrightarrow \mathrm{fib}(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{cofib}(f) \longrightarrow \Sigma X \longrightarrow \dots$$

then taking left adjoint twice in the above chain corresponds to projecting on the term to the right by one (and right adjoints go left). Applied to $f : \mathbb{S} \to \mathbb{S}$, which corepresents s, it also holds that taking left adjoint twice corresponds to looking at the functor corepresented by the next arrow to the left (and right adjoints go to the right now). This describes the entire chain since the left adjoint of a corepresentable functor is just given by tensoring with the corepresentable.

Remark 1.4.31 Smooth and proper categories are precisely the compact idempotent-complete stable categories which are enriched in compact spectra, hence they are in the sense the correct categorification of compact spectra. A similar result holds for finite-smooth and proper, as we now explain.

It is a result of Efimov, which also follows from Lurie's generalized Wall obstruction (see [Ram25] for an account of this), that a compact stable category is in the smallest subcategory of $\mathbf{Cat}^{\mathrm{Ex}}$ closed under pushouts, direct sums and containing $\mathrm{Sp}^{\mathrm{fin}}$ and $\mathrm{Ar}(\mathrm{Sp}^{\mathrm{fin}})$, if and only it is finite-smooth.

In particular, finite-smooth and proper subcategories are those compact categories which are both finite and enriched in finite spectra, hence the correct categorification of finite spectra (which, naturally are also the compact spectra, but very much not here).

2 Topological Hochschild homology of stable categories

2.1 A plethora of formulas

Fix $(\mathcal{C}, \otimes)$ a symmetric monoidal category and $X \in \mathcal{C}$ dualizable.

Definition 2.1.1 Let $M : X \to X$ be an endomorphism of X, then the *trace of* M is the endomorphism of the unit

$$\mathrm{tr}(M) : \mathbb{1} \xrightarrow{\mathrm{coev}} X^{\vee} \otimes X \xrightarrow{\tau \circ (\mathrm{id} \otimes M)} X \otimes X^{\vee} \xrightarrow{\mathrm{ev}} \mathbb{1}$$

In particular, in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$, traces of endomorphisms are colimit-preserving functors $\mathrm{Sp} \to \mathrm{Sp}$, i.e. just a point in Sp. For $\mathbf{Cat}^{\mathrm{Ex}}$ and $\mathbf{Cat}^{\mathrm{perf}}$, it is a point in Sp^{ω}.

Definition 2.1.2 Let $\mathcal{C}$ be a compactly-generated category and $M \in \mathrm{End}^{\mathrm{L}}(\mathcal{C})$, then we let:

$$\mathrm{THH}(\mathcal{C}, M) := \mathrm{tr}(M)$$

and we call this spectrum the *topological Hochschild homology of* $\mathcal{C}$ *with coefficients in* M.

Of course, the above definition makes more generally sense for endofunctors of dualizable categories. Actually, we will see later that THH is *localizing*, Efimov's unique extension applies and at least for coefficients in the identity, we would have recovered the extension for free. Note also the following consequence of Corollary 1.4.26.

Corollary 2.1.3 If $\mathcal{C}$ is smooth and proper and $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ is a colimit-preserving functor who also preserves compact objects, then the spectrum $\mathrm{THH}(\mathrm{Ind}(\mathcal{C}), M)$ is compact.

Remark that $\mathrm{End}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{C}))$ identifies with $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C})$, either by using that $\mathrm{Ind}(\mathcal{C})$ is dualizable or directly by currying, so that the explicit equivalence sends $M \in \mathrm{End}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{C}))$ to the functor exact in each variable $\mathrm{map}(-, M(-))$.

Thanks to the proof of Proposition 1.4.23, we therefore have:

Corollary 2.1.4 Let $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ be colimit-preserving, then

$$\mathrm{THH}(\mathcal{C}, M) \simeq \int^{X \in \mathcal{C}} \mathrm{map}(X, M(X))$$

In particular, if $\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{Sp}$, then $\mathrm{THH}(\mathrm{Sp}^{\mathrm{fin}}, M) \simeq M$ where we identified the colimit-preserving $M : \mathrm{Sp} \to \mathrm{Sp}$ with its value at the generator $\mathbb{S}$.

Let us make the following notational comment: we will often directly consider endomorphisms of $\mathrm{Ind}(\mathcal{C})$ as functors $M : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ which are exact in both variables, and we write $\mathrm{THH}(\mathcal{C}, M)$ for the coend of M, although we note that this spectrum only really depends on $\mathrm{Ind}(\mathcal{C})$ or equivalently, its compact objects $\mathrm{Idem}(\mathcal{C})$. Note in particular the following Corollary:

Corollary 2.1.5 Let $\mathcal{C}$ be a stable category and M a $\mathcal{C}$-bimodule. Then,

$$\mathrm{THH}(\mathcal{C}, M) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{Idem}(\mathcal{C}), M)$$

is an equivalence.

On a different "fun" direction, we also note that the formula of Corollary 2.1.4 is very similar to that of the very concrete traces of endomorphisms of finite dimensional vector spaces (up to turning a sum into an integral, which is fortunately the notation for coends). In particular, the coend that defines $\mathrm{THH}(\mathcal{C}, M)$ implies that points of the associated infinite loop-space should correspond to formal combination of maps $X \to M(X)$ subject to some identifications along maps $X \to Y$. We want now to be able to be more precise than this rough sketch, and for this we will first need a technical input.

The technical input is the following formula, often known as the Bousfield–Kan formula, which shows that any colimit decomposes as a geometric realization followed by a space-indexed colimit. We will not fully prove this — the main reason is model-theoretic difficulties that can probably be overcome but not without a hefty effort — but at least we attempt to explain parts of the proof; the following result is Corollary 12.5 of [Sha23] which explains a complete proof (but see also §5 of [MG19] for a differently worded proof). We follow the account of [Hau21], skipping the details related to the combinatorics of Δ.

Proposition 2.1.6 — Bousfield–Kan formula. Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$ with target a cocomplete category. Let $T_n := \mathrm{Map}(\Delta^n, \mathcal{C})$ so that the inclusion $T_0 \simeq \mathcal{C}^{\simeq} \to \mathcal{C}$ yields a composite

$$\phi_n : T_n \xrightarrow{d_0} T_0 \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

Then, the $\mathrm{colim}_n\, \phi_n$ assemble into a simplicial object as n varies whose geometric realisation is colim F. In formula, this reads:

$$\operatorname{colim} F \simeq \left| \ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \operatorname*{colim}_{(X_0 \to X_1) \in \operatorname{Fun}(\Delta^1, \mathcal{C})^{\simeq}} F(X_0) \rightrightarrows \operatorname*{colim}_{X_0 \in \mathcal{C}^{\simeq}} F(X_0) \right|$$

Sketch of proof. Let us rephrase the statement more formally: consider $\Delta_{/\mathcal{C}}$, the subcategory of $\mathbf{Cat}_{/\mathcal{C}}$ obtained by pulling along $\Delta \to \mathbf{Cat}$. This category is the total space of right fibration over Δ which classifies the functor $T : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ whose values are $T_n := \operatorname{Map}(\Delta^n, \mathcal{C})$. Because categories are in particular complete Segal objects in $\mathcal{S}$, note that $\mathcal{C}$ is the colimit of the composite $\lambda : \Delta_{/\mathcal{C}} \to \Delta \to \mathbf{Cat}$, where the map $\Delta \to \mathbf{Cat}$ is the usual inclusion.

By Lemma 1.1.2, $\mathcal{C}$ is therefore the localisation of the unstraightening of λ. As λ is a composite, we see that this unstraightening is of the form $\Delta_{/\mathcal{C}} \times_{\Delta} \Delta_{*/}$. We can now toy with the combinatorics of Δ to see that it suffices to invert cocartesian edges over maps in $\Delta_{/\mathcal{C}}$ whose component in Δ is of the form $d_0 : [n] \to [0]$. This in turn removes the need to pointify, so that $\mathcal{C}$ is also the localisation of $\Delta_{/\mathcal{C}}$ at the above described collection of arrows.

In consequence, there is a functor

$$\alpha : \Delta_{/\mathcal{C}} \longrightarrow \mathcal{C}$$

which is both coinitial and cofinal by virtue of being a localisation. But Δ has an automorphism which reverses the order of every subset (this is nothing else than the restriction of $(-)^{\mathrm{op}}$ to this subcategory of $\mathbf{Cat}$). This leads to a different choice of edges to invert and therefore to a different value of the localisation, namely it is now given by $\mathcal{C}^{\mathrm{op}}$:

$$\beta : \Delta_{/\mathcal{C}} \longrightarrow \mathcal{C}^{\mathrm{op}}$$

By cofinality, the colimit of $F : \mathcal{C} \to \mathcal{D}$ is also the colimit of the composite $\Delta^{\mathrm{op}}_{/\mathcal{C}} \to \mathcal{C} \to \mathcal{D}$, which is computed by left Kan extension along $\Delta^{\mathrm{op}}_{/\mathcal{C}} \to \Delta^{\mathrm{op}} \to *$, which we can do in two steps, first along $\Delta^{\mathrm{op}}_{/\mathcal{C}} \to \Delta^{\mathrm{op}}$ which is taking the colimits of what was called ϕ_n in the statement of the theorem and then the geometric realization, i.e. left Kan extending along $\Delta \to *$. This concludes. □

In fact, by inserting in the proof that spaces are nothing more than geometric realizations of sets (their simplices), we also have a version of the above formula where the space-indexed colimits have become set-indexed, i.e. coproducts:

Proposition 2.1.7 — Also the Bousfield–Kan formula. Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$ with target a cocomplete category, and suppose $\mathcal{C}$ is explicitly given by a small simplicial set. Then, there is an equivalence

$$\operatorname{colim} F \simeq \left| \ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \coprod_{\alpha : X_0 \to X_1 \in \mathcal{C}_1} F(X_0) \rightrightarrows \coprod_{x \in \mathcal{C}_0} F(X_0) \right|$$

where the n-simplices of the right hand side is given by the coproduct over $\sigma \in \mathcal{C}_n$ of F evaluated at the first vertex of σ.

Let us now try to apply this formula to the coend defining THH. Recall that n-chains of the twisted arrow category of $\mathcal{C}$, i.e. functors $X : [n] \to \operatorname{TwAr}(\mathcal{C})$ are given by diagrams in $\mathcal{C}$ of the form:

$$\begin{array}{ccccccc} X_n & \longleftarrow & \ldots & \longleftarrow & X_1 & \longleftarrow & X_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_n & \longrightarrow & \ldots & \longrightarrow & Y_1 & \longrightarrow & Y_0 \end{array}$$

with $X_i \to Y_i$ being the value $X(i)$. In particular, we see clearly that $\operatorname{Fun}([n], \operatorname{TwAr}(\mathcal{C})) \simeq \operatorname{Fun}([n] \star [n]^{\mathrm{op}}, \mathcal{C})$ i.e. chains of length $2n+1$ obtained by forgetting all of the vertical maps except for the leftmost one.

In consequence, if $M : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ is biexact, we get the following formula:

$$\int^{X \in \mathcal{C}} M(X, X) \left| \ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \operatorname*{colim}_{X \in T_1} M(Y_0, X_0) \rightrightarrows \operatorname*{colim}_{X \in T_0} M(Y_0, X_0) \right|$$

where the colimits are indexed by $T_n \simeq \operatorname{Fun}([n], \operatorname{TwAr}(\mathcal{C}))^{\simeq}$ with the aforementioned notational convention.

The functor $e : \Delta \to \Delta$, sending $[n]$ to the join $[n] \star [n]^{\mathrm{op}} = [2n+1]$, is called the *edgewise subdivision*. As explained in section 2.2 of [Bar13], this functor is such that e^{op} is cofinal. In particular, the above geometric realization is equivalent to a simpler expression, with colimits indexed by spaces of length-n chains in $\mathcal{C}$:

Applied now to the coend that computes THH, we get:

Theorem 2.1.8 Let $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ be colimit-preserving, then, there is an equivalence

$$\mathrm{THH}(\mathcal{C}, M) \simeq \left| \dots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \operatorname*{colim}_{(X \to Y) \in \mathrm{Ar}(\mathcal{C})^{\simeq}} \mathrm{map}(Y, M(X)) \rightrightarrows \operatorname*{colim}_{X \in \mathcal{C}^{\simeq}} \mathrm{map}(X, M(X)) \right|$$

whose n-simplices is a colimit indexed by the space of length n chains $X_0 \to \dots \to X_n$ of the spectrum

$$\mathrm{map}(X_n, M(X_0))$$

Moreover, the formula can be further unravelled to show:

$$\mathrm{THH}(\mathcal{C}, M) \simeq \left| \dots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \operatorname*{colim}_{X,Y \in \mathcal{C}^{\simeq}} \mathrm{map}(X, Y) \otimes \mathrm{map}(Y, M(X)) \rightrightarrows \operatorname*{colim}_{X \in \mathcal{C}^{\simeq}} \mathrm{map}(X, M(X)) \right|$$

with n-simplicies the colimit indexed by tuples of $(X_0, \dots, X_n)$ of

$$\mathrm{map}(X_0, X_1) \otimes \dots \otimes \mathrm{map}(X_n, M(X_0))$$

Proof. The first formula we have already proven in the above discussion, so let us solely focus on deriving the second formula. Let us mention that one proof of this, which is certainly the most straightforward, is to directly check that the wanted formula, as a functor in M, constitutes a valid evaluation map $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \to \mathrm{Sp}$ hence must coincide with the coend by the unicity of such datum. This is how this formula is proven in [HSS17, Section 4.5].

Of course, the above miracle is unenlightening; it is possible to tweak the simplicial object a bit so that it no longer computes the trace and yet the phenomenon at heart of the above proof will still hold. The *more* correct way to prove the second formula is to appeal to two facts: Lemma **??**, which we will prove later once we have developed enough enriched category theory, and a spectrally-enriched version of the Bousfield–Kan formula, see Theorem **??**. In what follows, we therefore do not provide a proof but at least, we hope to explain where the subtlety lies, and what has to be done.

Let us begin by noting that $\mathrm{Fun}([n], \mathcal{C})^{\simeq}$ is the total space of a fibration over $\mathcal{C}^{\simeq} \times \dots \times \mathcal{C}^{\simeq}$ with $(n+1)$ factors whose fibers are the products of mapping spaces in successive order, i.e.

$$\mathrm{Map}(X_0, X_1) \times \dots \times \mathrm{Map}(X_{n-1}, X_n)$$

The colimit in each simplicial degree of the first formula only depends of the base; using that colimits over constant spaces are also given by tensoring with the suspension spectrum, we get the following formula

$$\mathrm{THH}(\mathcal{C}, M) \simeq \left| \dots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \operatorname*{colim}_{X,Y \in \mathcal{C}^{\simeq}} \Sigma^{\infty}_{+}\Omega^{\infty} \mathrm{map}(X, Y) \otimes \mathrm{map}(Y, M(X)) \rightrightarrows \operatorname*{colim}_{X \in \mathcal{C}^{\simeq}} \mathrm{map}(X, M(X)) \right|$$

where the general term is given a colimit over tuples of X_i of

$$\Sigma^{\infty}_{+}\Omega^{\infty} \mathrm{map}(X_0, X_1) \otimes \dots \otimes \Sigma^{\infty}_{+}\Omega^{\infty} \mathrm{map}(X_{n-1}, X_n) \otimes \mathrm{map}(X_n, M(X_0))$$

Recall that for F an exact functor, the exact approximation of $\Sigma^{\infty}_{+}\Omega^{\infty} F$ is F itself; in particular there is a natural transformation $\Sigma^{\infty}_{+} \mathrm{Map} \to \mathrm{map}$ which witness the target as the exact approximation of the source.

To conclude, we would have to adapt the idea that underlies Remark 2.1.12 and be able to replace each $\Sigma^{\infty}_{+}\Omega^{\infty} \mathrm{map}(X_{n-1}, X_n)$ by its exact approximation $\mathrm{map}(X_{n-1}, X_n)$. Let us mention that is nothing short of a miracle, at least from the prism of enriched category theory: indeed, we will eventually interpret this fact as saying that coend of exact functors are automatically spectrally-enriched, but a similar fact is *not* true for general $\mathcal{V}$-enriched categories. □

The above formula is fun and arguably more natural, because it uses the Bousfield–Kan formula involving colimits of spaces, which sounds more reasonable to the homotopy theorist. But there is another, which has for itself the favours of history:

Proposition 2.1.9 Let $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ be colimit-preserving, then, there is an equivalence

$$\mathrm{THH}(\mathcal{C}, M) \simeq \left| \dots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{X,Y\in\mathcal{C}} \mathrm{map}(X,Y)\otimes \mathrm{map}(Y, M(X)) \substack{\longrightarrow\\\longrightarrow} \coprod_{X\in\mathcal{C}} \mathrm{map}(X, M(X)) \right|$$

whose n-simplices is a colimit indexed by the space of tuples $X_0, \dots, X_n$ of the spectrum

$$\mathrm{map}(X_0, X_1)\otimes \dots \otimes \mathrm{map}(X_n, M(X_0))$$

Proof. The whole previous discussion also adapts with the Bousfield–Kan formula of Proposition 2.1.7, we do not redo it. □

The formula of Proposition 2.1.9 has sometimes be termed as the *multi-object (cyclic) Bar construction*, the cyclic part coming when $M \simeq \mathrm{map}$. We will explain in the next section the appearance of the term *cyclic*; let us also make a remark about the term *multi-object*.

Remark 2.1.10 Let us make here a long historical remark. For a discrete ring R, $\mathrm{THH}(R)$ was first defined by Bökstedt in [Bö85] via a formula as in Proposition 2.1.9 except over $\mathcal{C}$ the category with one object and R as its endomorphism spectrum (so a spectrally-enriched category, though he did not put it in these words). We will compare this definition with ours later; let us also mention the fact that Bökstedt was deeply inspired by Waldhausen's insights in this construction and, relevant to this lecture because of where it is given, many of these early developments happened in Bielefeld.

For a ring $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$, Bökstedt formula still works and one can even generalize a little. Indeed, a R-bimodule M induces a functor $M \otimes_R - : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$, and reciprocally any such colimit-preserving functor is of this form, since $\mathrm{Mod}(R)$ is (non-freely) generated under colimits by a single object, namely R viewed as a bimodule over itself. In this context, [NS17, KMN23] have produced cyclic Bar constructions refining the above, but again, they are not quite the same, as they look as follows:

$$\mathrm{THH}(R, M) := \left| \dots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} R\otimes M \substack{\longrightarrow\\\longrightarrow} M \right|$$

where the tensor product is absolute, i.e. over $\mathbb{S}$. Again, it turns out that properly explaining this equivalence of definitions involves ideas that we are not ready to explain, so we delay this to a later section.

Finally, we also have to mention that in [DM96], there is also a cyclic Bar formula with many objects for *ring functors*. It is unclear to the author, due to a lack of background into that specific era of trace methods mathematics, what actually a ring functor is, but sources tell him it should be thought as a way of encoding spectrally-enriched categories, so related to the material we will be developing later.

In any case, they specialize their theory to additive categories for ungodly model categorical reasons (i.e. their THH is not in general an Ω-spectrum), and if R is a ring or in fact, a connective ring spectrum, compare their $\mathrm{THH}(\mathrm{Proj}(R))$ to Bökstedt's $\mathrm{THH}(R)$, where $\mathrm{Proj}(R)$ is the smallest subcategory of $\mathrm{Perf}(R)$ containing R and closed under retracts and extensions. In particular, since R is connective, $\mathrm{Perf}(R)$ is the stable envelope of $\mathrm{Proj}(R)$ and so admitting the above comparison, we will be able to compare Dundas–McCarthy's and our construction in a simpler way, but again in later developments.

In view of the above lengthy historical remark (which is also omitting some constructions, such as [DM94], which is another paper that exists thanks to the welcoming Universität Bielefeld), we hope that the ultimate goal of this section is more motivated: provide a universal property for THH.

However, let us conclude this section by applying the Bousfield–Kan formula to the stable coend Yoneda lemma we proved in 1.4.22.

Corollary 2.1.11 Let $\mathcal{C}$ be stable and $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ exact, then

$$F(A) \simeq \left| \dots \rightrightarrows \operatorname*{colim}_{X \to Y \in \mathrm{Ar}(\mathcal{C})^{\simeq}} F(Y) \otimes \mathrm{map}(A, X) \rightrightarrows \operatorname*{colim}_{X \in \mathcal{C}^{\simeq}} F(X) \otimes \mathrm{map}(A, X) \right|$$

and more generally, the n-simplicies are given by

$$\operatorname*{colim}_{(X_0 \to \dots \to X_n) \in \mathrm{Fun}([n], \mathcal{C})^{\simeq}} F(X_n) \otimes \mathrm{map}(A, X_0)$$

with simplicial maps induced by precomposition.

There is also a similar formula with coproducts in place of colimits by replacing Proposition 2.1.6 by Proposition 2.1.7.

Remark 2.1.12 Let us also note that the right hand side formula is always exact in A, and actually computes the initial functor under F which is exact. This follows from checking already that the natural transformation

$$F(-) \longmapsto \int^{X \in \mathcal{C}} F(X) \otimes \mathrm{map}_{\mathcal{C}}(-, X)$$

has the wanted property. But by the coend Yoneda Lemma, this map is an equivalence for exact presheaves and its target is always exact (because map is) so the local criterion for Bousfield localisations concludes ([Lur08, Proposition 5.2.7.4]). This has the property that for computing the above coend, one can replace F by its exact approximation.

2.2 Two universal characterizations

So far, we have explored a lot about the spectrum $\mathrm{THH}(\mathcal{C}, M)$ but we have been silent about the functor. The goal of this section is to remedy this, and to further study THH by giving two characterizations of the functor: universality with respect to a "weak" set of properties and uniqueness with respect to a "strong" set of properties.

Let us begin by certainly the easier fact on functoriality:

Lemma 2.2.1 There is a functor $\mathrm{THH}(\mathcal{C}, -) : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}$.

Proof. This functor is the composite

$$\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp}) \longrightarrow \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{TwAr}(\mathcal{C}), \mathrm{Sp}) \xrightarrow{\mathrm{colim}} \mathrm{Sp}$$

where the first map is precomposition along $\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. □

From the coend formula perspective, the above is saying that a natural transformation $M \to N$ of $\mathcal{C}$-bimodules induces a map between their colimits. But given M a $\mathcal{D}$-bimodule and $f : \mathcal{C} \to \mathcal{D}$, we can define a $\mathcal{D}$-bimodule $M \circ (f^{\mathrm{op}} \times f)$. Then, there is a natural map

$$\operatorname*{colim}_{(X \to Y) \in \mathrm{TwAr}(\mathcal{C})} M(f(Y), f(X)) \longrightarrow \operatorname*{colim}_{(X' \to Y') \in \mathrm{TwAr}(\mathcal{D})} M(Y', X')$$

induced by restriction of the diagram via the induced map $\mathrm{TwAr}(f) : \mathrm{TwAr}(\mathcal{C}) \to \mathrm{TwAr}(\mathcal{D})$ hence there is also some functoriality between the different $\mathrm{THH}(\mathcal{C}, -)$.

Our goal is to explain how to bundle these into one single object. For this, we first have to bundle the source categories together. Note that the association

$$\mathcal{C} \longmapsto \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$$

is both covariantly functorial via left Kan extension along $f^{\mathrm{op}} \times f$ or contravariantly using precomposition along $f^{\mathrm{op}} \times f$. Since those two operations are adjoint to one another, we can give the following definition:

Definition 2.2.2 We write $\mathbf{Cat}^{\mathrm{lace}}$ for the total space of the bicartesian fibration which classifies the above functor and its two adjoint functorialities.

Note that $\mathbf{Cat}^{\mathrm{lace}}$ is also a full subcategory of a category whose objects are usually called the category of profunctors on $\mathbf{Cat}^{\mathrm{Ex}}$ (or sometimes distributors, or relators, or even bimmodules) spanned by profunctors with the same source and target. In fact the whole category of profunctors is encoded in $\mathbf{Cat}^{\mathrm{lace}}$: as we will see in Proposition 2.2.12, profunctors with different source and target show up as "adjoints arrows" on $\mathbf{Cat}^{\mathrm{lace}}$. This is a different take on profunctors, which does not involve any 2-categorical refinement a priori (but ultimately, the aforementioned Proposition and what we will call *laced semi-orthogonal decomposition* are 2-categorical) or in any case, intertwined the 2-categorical datum at the 1-categorical level, in such a way that makes it particularly comfortable to deal with topological Hochschild homology and as we will later see, cyclic K-theory.

Remark 2.2.3 Unstraightening is sometimes denoted by an integral sign, which we have chosen to avoid for it is our notation for (co)ends, but let us use it for this very Remark:

$$\mathbf{Cat}^{\mathrm{lace}} := \int^{\mathcal{C}\in\mathbf{Cat}^{\mathrm{Ex}}} \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}, \mathrm{Sp})$$

There is a great deal of similarity, half for arbitrary notational reasons but the other half for a good reason, between this formula and the coend formula for THH:

$$\mathrm{THH}(\mathcal{C}, M) := \int^{X\in\mathcal{C}} M(X,X)$$

We will see throughout this section that $\mathbf{Cat}^{\mathrm{lace}}$ is a very natural home for THH, and we would like to think about it as a *categorified* version of the coend formula for THH — or maybe more rightfully so a *lax-categorified* version (i.e. we will still need to impose a condition of inverting some maps to recover something closer to THH).

A surprising phenomenon, which is at the heart of [HNRS26a, HNRS26b], is that the lax categorifications of the different formulae for THH, as in Proposition 2.1.9, or its cyclic and epicyclic refinements, see later sections, lead to different lax-categorifications of THH, which are such that when enforcing the correct strictness condition, they either become equivalent or one is extra-structure of the other. If time permits, this is what these lecture notes would want to explain.

There is a second description which offers a different, but still particularly tractable description of $\mathbf{Cat}^{\mathrm{lace}}$:

Lemma 2.2.4 The category $\mathbf{Cat}^{\mathrm{lace}}$ fits in the following pullback square:

$$\begin{array}{ccc} \mathbf{Cat}^{\mathrm{lace}} & \longrightarrow & \mathrm{Ar}^{\mathrm{oplax}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathbf{Cat}^{\mathrm{Ex}} & \xrightarrow{(\mathrm{Ind},\mathrm{Ind})} & \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}} \times \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}} \end{array}$$

Proof. The pullback classifies the correct functor: indeed, it is a classical fact that $\mathrm{Ar}^{\mathrm{oplax}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}) \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}} \times \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ is the unstraightening of $\mathrm{Fun}^{\mathrm{L}}(-,-)$ made a functor by precomposing in one variable and postcomposing in the other. In particular, this is cocartesian in one variable and cartesian in the other, but since both operations have adjoints, turns out to be globally bicartesian. □

Explicitly, $\mathbf{Cat}^{\mathrm{lace}}$ is a category of pairs $(\mathcal{C}, M)$, called *laced categories*, with $M : \mathcal{C}^{\mathrm{op}}\otimes\mathcal{C} \to \mathrm{Sp}$. Maps $(\mathcal{C}, M) \to (\mathcal{D}, N)$ in this category, *laced functors*, are pairs $f : \mathcal{C} \to \mathcal{D}$ and a natural transformation

$$\eta : M \longrightarrow (f^{\mathrm{op}} \times f)^* N$$

or dually, from the left Kan extension of M. Let us also note that using $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}, \mathrm{Sp}) \simeq \mathrm{End}^{\mathrm{L}}(\mathrm{Ind}\,\mathcal{C})$, we can also write the above in term of pre- and postcomposition along $\mathrm{Ind}(f)$ and its right adjoint.

Lemma 2.2.5 The category $\mathbf{Cat}^{\mathrm{lace}}$ is presentable, in fact compactly-generated. It has a symmetric monoidal structure with unit $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$ which makes $\mathbf{Cat}^{\mathrm{lace}} \to \mathbf{Cat}^{\mathrm{Ex}}$ a symmetric monoidal functor; the symmetric monoidal structure is closed with internal mapping object:

$$\underline{\mathrm{Fun}}((\mathcal{C}, M), (\mathcal{D}, N)) := (\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}), \mathrm{Nat}_N^M)$$

where $\mathrm{Nat}_N^M(f, g) := \mathrm{Nat}(M, N \circ (f^{\mathrm{op}} \times g))$.

Proof. All of those facts are proven in [HNS24], from which we draw inspiration.

We first claim that $\mathbf{Cat}^{\mathrm{lace}}$ is compactly-generated by two objects $(\mathrm{Sp}^{\mathrm{fin}}, 0)$ and $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$. Now of course the former is a retract of the latter so in fact, $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$ is a single compact generator. Recall also that $(\mathrm{Sp}^{\mathrm{fin}}, 0)$ corepresents the functor $(\mathcal{C}, M) \mapsto \mathcal{C}^{\simeq}$ whereas $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$ corepresents $(\mathcal{C}, M) \mapsto \mathrm{Lace}(\mathcal{C}, M)^{\simeq}$. We will investigate later this functor in more details; the only thing we need is that $\mathrm{Lace}(\mathcal{C}, M)^{\simeq} \to \mathcal{C}^{\simeq}$ is a left fibration classifying $X \mapsto \Omega^\infty M(X, X)$. To see this, it suffices to see that there is a pullback square

$$\begin{array}{ccc} \mathrm{Lace}(\mathcal{C}, M)^{\simeq} & \longrightarrow & \mathrm{Ar}(\mathrm{Ind}(\mathcal{C}))^{\simeq} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\simeq} & \xrightarrow{(j, M)} & \mathrm{Ind}(\mathcal{C})^{\simeq} \times \mathrm{Ind}(\mathcal{C})^{\simeq} \end{array}$$

which follows from the pullback square of Lemma 2.2.4. The corresponding fact for the right hand vertical map implies the claim about the left hand side vertical map.

Given $f : (\mathcal{C}, M) \to (\mathcal{D}, N)$, there is a commutative square

$$\begin{array}{ccc} \mathrm{Lace}(\mathcal{C}, M)^{\simeq} & \longrightarrow & \mathrm{Lace}(\mathcal{D}, N)^{\simeq} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\simeq} & \longrightarrow & \mathcal{D}^{\simeq} \end{array}$$

whose vertical legs are left fibrations classifying respectively $X \mapsto \Omega^\infty M(X, X)$ and $X \mapsto \Omega^\infty N(X, X)$. Hence, if both horizontal arrows are equivalences, then so is at every point the natural map $\Omega^\infty M(X, X) \to \Omega^\infty N(f(X), f(X))$. In consequence, the natural transformation

$$\eta : M(-, -) \longrightarrow N(f(-), f(-))$$

is itself an equivalence. But note that $f : \mathcal{C} \to \mathcal{D}$ is already an equivalence by Corollary 1.3.10 so that the above laced functor (f, η) is itself an equivalence.

Again, we skip the details for the symmetric monoidal structure: it follows from the cartesian square of Lemma 2.2.4. Let us therefore only describe what it does on objects: $(\mathcal{C}, M) \otimes (\mathcal{D}, N) := (\mathcal{C} \otimes \mathcal{D}, M \boxtimes N)$ where $M \boxtimes N$ is the unique $\mathcal{C} \otimes \mathcal{D}$ bimodule induced by the bi-exact functor $M \times N$.

Finally, the formula for the internal mapping object can also directly be checked from the cartesian square, though this is cumbersome to spell out. ☐

Remark 2.2.6 Let us also mention that $\mathbf{Cat}^{\mathrm{lace}}$ can also be identified with $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, the tangent bundle of $\mathbf{Cat}^{\mathrm{Ex}}$. This means that the forgetful functor $\mathbf{Cat}^{\mathrm{lace}} \to \mathbf{Cat}^{\mathrm{Ex}}$ is a cartesian fibration which classifies the functor $\mathcal{C} \mapsto \mathrm{Sp}(\mathbf{Cat}^{\mathrm{Ex}}_{/\mathcal{C}})$ — after the fact, the identification $\mathrm{Sp}(\mathbf{Cat}^{\mathrm{Ex}}_{/\mathcal{C}}) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$ implies this is actually a cocartesian fibration as well, since the restriction functors have left adjoints given by left Kan extension.

We could check that THH extends to a functor on $\mathbf{Cat}^{\mathrm{lace}}$ by hand, but that would not be as pretty and as formal as what we will do instead. We want to understand how much of the properties of a *trace* THH has retained; for this, let us introduce an intermediate object:

Definition 2.2.7 We write $\mathbf{Cat}^{\mathrm{unlace}}$ for the unstraightening of the functor $\mathcal{C} \mapsto \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$, with either covariant functoriality given by left Kan extension or contravariant functoriality given by pullbacks.

Warning 2.2.8 By a theorem of Harpaz–Nuiten–Prasma, see [HNP18, Theorem 1.0.3], $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Sp})$ is *not* the tangent bundle of **Cat** at $\mathcal{C}$ so even if we had tried to compensate the above lack of stability, the analogue of Remark 2.2.6 fails.

The actual tangent bundle is $\mathrm{Fun}(\mathrm{TwAr}(\mathcal{C}), \mathrm{Sp})$ — there is a fun little coincidence, since $\mathrm{TwAr}(\mathcal{C})$ is the category we are taking colimits over when writing the coends that define THH. It is unclear to the author whether there is more to it than this coincidence. See also Remark **??** for related remark on the general, $\mathcal{V}$-enriched case.

The inclusion $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$ induces a functor $\mathbf{Cat}^{\mathrm{lace}} \to \mathbf{Cat}^{\mathrm{unlace}}$.

Proposition 2.2.9 The category $\mathbf{Cat}^{\mathrm{lace}}$ is tensored and cotensored over $\mathbf{Cat}^{\mathrm{unlace}}$.

Proof. The category $\mathbf{Cat}^{\mathrm{unlace}}$ fits in a pullback square similar to that of Lemma 2.2.4:

$$\begin{array}{ccc} \mathbf{Cat}^{\mathrm{unlace}} & \longrightarrow & \mathrm{Ar}^{\mathrm{oplax}}(\mathrm{Pr}^{\mathrm{L}}) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathbf{Cat} & \xrightarrow{(\mathcal{P},\mathcal{P})} & \mathrm{Pr}^{\mathrm{L}} \times \mathrm{Pr}^{\mathrm{L}} \end{array}$$

In particular, the above natural transformation is also induced by a natural transformation between the cospans, induced by forgetting the stability. But note that all those functors have a left adjoint, given by some flavour of stabilization: either the Spanier-Whitehead stabilization of freely adding finite colimits to a category for small categories or simply tensoring with Sp for presentable ones. This induces a left adjoint to the forgetful functor $\mathbf{Cat}^{\mathrm{lace}} \to \mathbf{Cat}^{\mathrm{unlace}}$, given in formula by:

$$(\mathcal{C}, I) \longmapsto \mathrm{St}(\mathcal{C}, I) := (\mathrm{SW}(\mathcal{P}^{fin}_{\emptyset}(\mathcal{C})), \widetilde{\Sigma^{\infty}_{+} I})$$

where we borrowed the notation of Theorem 1.2.2 and $\widetilde{\Sigma^{\infty}_{+} I}$ denotes the unique extension of $\Sigma^{\infty}_{+} I : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ to a $\mathrm{St}(\mathcal{C})$-bimodule.

But combining Lemma 1.4.7 and Lemma 1.4.1, we see that the left adjoints in the map of cospans are symmetric monoidal functors. Ergo, since the symmetric monoidal structures on $\mathbf{Cat}^{\mathrm{unlace}}$ and $\mathbf{Cat}^{\mathrm{lace}}$ are built via the pullback, it follows that $\mathbf{Cat}^{\mathrm{lace}}$ is tensored and cotensored over $\mathbf{Cat}^{\mathrm{unlace}}$ and these operations factor through the left adjoint St. □

In particular, we will be interested in the following example: $(\mathcal{C}, M)^{([n],*)}$, that is the cotensor of a laced category $(\mathcal{C}, M)$ by the unstable laced category $([n], *)$, where $[n] := \{0 < ... < n\}$ and the bimodule is constant equal to the point. This cotensor is a priori the laced category:

$$(\mathcal{C}, M)^{([n],*)} := \underline{\mathrm{Fun}}(\mathrm{St}([n], *), (\mathcal{C}, M)) \simeq (\mathrm{Fun}([n], \mathcal{C}), \mathrm{Nat}^*_N)$$

where the associated bimodule Nat^*_N is given by $\mathrm{Nat}^*_N(f, g) = \mathrm{nat}(\mathrm{cst}_{\mathbb{S}}, M)$. Using the description of natural transformations as an end, as in Lemma 1.4.20, we can compute it more explicitly as the following, coming from evaluating at the initial object of $\mathrm{TwAr}([n])$:

$$M^{([n],*)}(X_0 \to ... \to X_n, Y_0 \to ... \to Y_n) := M(X_n, Y_0)$$

Because of the functoriality of cotensors, for a fixed laced category $(\mathcal{C}, M)$, there is a simplicial laced category $(\mathcal{C}, M)^{([\bullet],*)}$ and this construction is also functorial in $(\mathcal{C}, M)$, i.e. defines a functor

$$\mathbf{Cat}^{\mathrm{lace}} \times \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}^{\mathrm{lace}}$$

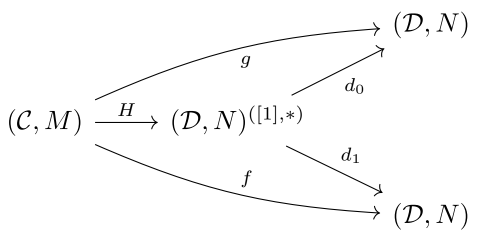
Definition 2.2.10 A functor $\mathbf{Cat}^{\mathrm{lace}} \to \mathcal{E}$ is *trace-like* if it inverts all the maps in the simplicial object $(\mathcal{C}, M)^{([\bullet],*)}$ for every $(\mathcal{C}, M)$.

Because of the simplicial relations, it suffices to check that such a functor inverts one exterior face of every degree, and among such faces, it suffices to check that say every $d_0 : (\mathcal{C}, M)^{([1],*)} \to$

$(\mathcal{C}, M)$ is inverted by using that this implies that $(\mathcal{C}, M)^{([n]\times[1],*)} \to (\mathcal{C}, M)^{([n],*)}$ also will be inverted.

There are many more maps inverted by any trace-like functors than just the ones in the simplicial object. The following is a classical idea in simplicial homotopy theory which provides a bigger class of inverted maps, though it is unclear to the author whether this is all of them.

Lemma 2.2.11 Suppose given a diagram of the following form in $\mathbf{Cat}^{\text{lace}}$:



Then, any trace-like functor inverts f if and only if it inverts g.

Proof. This is straightforward: if $\Phi : \mathbf{Cat}^{\text{lace}} \to \mathcal{E}$ is trace-like and inverts say f, then it must invert the *homotopy* H because it inverts d_0, d_1 by assumption. Considering the second triangle immediately concludes. ☐

In particular, if a pair of functor f, g is such that there exists two diagrams as above between $f \circ g$ and id and respectively $g \circ f$ and id in the second diagram, then both f, g are inverted by any trace-like functor. The name *trace-like* might be a bit coming out of the blue given the above definition, but the following might be helpful in that regard.

Proposition 2.2.12 Let $L : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : R$ be an adjunction between stable categories. Let $M : \mathcal{D}^{\text{op}} \otimes \mathcal{C} \to \text{Sp}$ be an exact functor. Then, there is a pair of laced functors

$$L_M : \ (\mathcal{C}, M \circ (L^{\text{op}} \times \text{id})) \rightleftarrows (\mathcal{D}, M \circ (\text{id} \times R)) \ : R_M$$

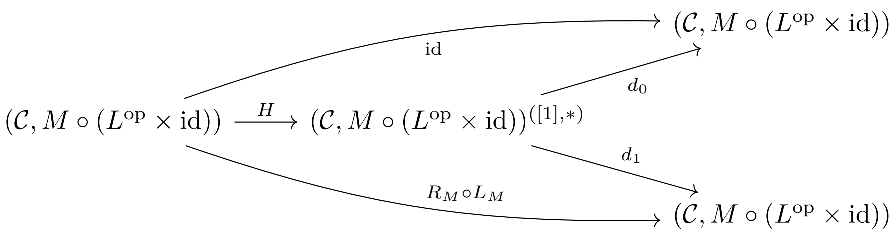
lifting the adjunction $L \dashv R$ which are sent to maps inverse to one another by every trace-like functor.

Proof. To define the laced functors, it suffices to produce a natural transformation for each, respectively

$$M \circ (L^{\text{op}} \times \text{id}) \longrightarrow M \circ (\text{id} \times R) \circ (L^{\text{op}} \times L)$$

$$M \circ (\text{id} \times R) \longrightarrow M \circ (L^{\text{op}} \times \text{id}) \circ (R^{\text{op}} \times R)$$

and we take the maps induced by the unit and the counit. But note there is the following commutative diagram:



where H simply witnessed that the unit of the adjunction comes with a laced refinement by the above natural transformation. A dual diagram reversing the order of L_M and R_M and using the counit finishes the proof. ☐

Remark 2.2.13 In the convention of bimodules as Ind-functors, the above pair of laced functors reads

$$(\mathcal{C}, \mathrm{Ind}(R) \circ M) \rightleftarrows (\mathcal{D}, M \circ \mathrm{Ind}(R))$$

where M is viewed as a functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$. Hence, trace-like functors have the cyclic invariance characteristic of a trace when one of the bimodule is a functor $\mathrm{Ind}\,\mathcal{C} \to \mathrm{Ind}\,\mathcal{D}$ which preserves compact objects and whose adjoint also preserves compact objects.

If $\mathcal{C}, \mathcal{D}$ are categories of modules over a ring, this corresponds to those bimodules M which are compact and whose dual is also compact. More precisely, this means that given a compact (S, R)-module N, in particular N compact as S module, and writing $L := N \otimes_R - : \mathrm{Perf}(R) \to \mathrm{Perf}(S)$, and also given M any (R, S)-bimodule then there is an equivalence

$$F(\mathrm{Perf}(R), N \otimes_R M) \simeq F(\mathrm{Perf}(S), M \otimes_R N)$$

coming from explicit maps in $\mathbf{Cat}^{\mathrm{lace}}$.

Given our definition, it is not too hard to produce the universal trace-like functor out of a given F. In particular, the following strategy of proof is quite robust to other situations:

Proposition 2.2.14 Let $F : \mathbf{Cat}^{\mathrm{lace}} \to \mathcal{E}$ be a functor to category with geometric realizations, then the functor $\mathrm{cyc}(F)$ given by

$$\mathrm{cyc}(F)(\mathcal{C}, M) := \left| F((\mathcal{C}, M)^{([\bullet],*)}) \right|$$

is trace-like and receives a natural transformation $F \to \mathrm{cyc}(F)$ which is initial among natural transformations with source F and target a trace-like functor.

Proof. The construction cyc is an augmented endofunctor, the augmentation $\eta : \mathrm{id} \implies \mathrm{cyc}$ being induced from inclusion of 0-simplices. Therefore it suffices to check the criterion of 5.2.7.4(3) of [Lur08].

We first claim the essential image of cyc is precisely trace-like functors. Indeed, if F is trace-like, $\mathrm{cyc}(F)$ is constant hence $F \to \mathrm{cyc}(F)$ is an equivalence. Moreover, remark that

$$\mathrm{cyc}(F)((\mathcal{C}, M)^{([1],*)})) := \left| F((\mathcal{C}, M)^{([\bullet]\times[1],*)}) \right|$$

is always equivalent to $\mathrm{cyc}(F)(\mathcal{C}, M)$ via either projections. This is a general fact about functors defined on the (opposite of the) product-closure of Δ: there is always an equivalence $|X([\bullet]\times[1])| \simeq |X([\bullet])|$ — this readily follows from the fact that $(\mathrm{id}, \mathrm{cst}_{[k]}) : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal for every $[k] \in \Delta$. In particular, this showcases the usefulness of having generally defined cotensors.

Finally, to conclude, remark that $\mathrm{cyc}(\mathrm{cyc}(F))$ is the colimit of a bisimplicial object which is constant in either simplicial direction, hence the two maps we have to check are invertible for the criterion are evidently equivalences as inclusion of 0-simplicies in either simplicial direction. □

We claim one can identify THH as cyc applied to a specific functor. There is no doubt to which functor this is supposed to be, as F is always the 0-simplices of this simplicial object — however, we have produced multiple geometric realizations formulae for THH, so this is where we have to be smart. We will pick one we have skipped explaining using the coend Yoneda lemma.

If $(\mathcal{C}, M)$ is a laced-category, let us define the *naive trace* of $(\mathcal{C}, M)$ as follows:

$$\widehat{\mathrm{tr}}(\mathcal{C}, M) := \operatorname*{colim}_{X \in \mathcal{C}^{\simeq}} M(X, X)$$

The notation $\widehat{\mathrm{tr}}$ is trying to imitate the trace wearing a hat too big for itself (I did not want to spend the time to make it slanted, but it probably should). Note that this is indeed a functor not just in M but in $(\mathcal{C}, M) \in \mathbf{Cat}^{\mathrm{lace}}$, namely the composite

$$\mathbf{Cat}^{\mathrm{lace}} \longrightarrow \int_{\mathcal{C}\in\mathbf{Cat}^{\mathrm{Ex}}} \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}, \mathrm{Sp}) \xrightarrow{\text{“colim”}} \int_{\mathcal{C}\in\mathbf{Cat}^{\mathrm{Ex}}} \mathrm{Sp} \longrightarrow \mathrm{Sp}$$

where the integral sign is a notation for the unstraightening. The second-to-last term is equivalently $\mathbf{Cat}^{\mathrm{Ex}} \times \mathrm{Sp}$ so that the last map is the projection to Sp. The other two maps are maps of cocartesian fibrations, the former relies on the compatibility of the functoriality with restriction along $\mathcal{C}^{\simeq}\times\mathcal{C}^{\simeq}\to\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$. The latter is essentially due to the (lax) compatibility with colimits (in particular, it is not a map that respects cocartesian lifts).

Another approach to define this functor is to first build the functor sending $(\mathcal{C},M)$ to

$$\operatorname*{colim}_{X\in\mathcal{C}^{\simeq}}\Omega^\infty M(X,X)$$

but note that this is another name for $\mathrm{Lace}(\mathcal{C},M)^{\simeq}$, since this is the total space of the fibration classifying this functor. In particular, this is clearly a functor in $\mathbf{Cat}^{\mathrm{lace}}$. Then, $\widehat{\mathrm{tr}}$ is the target of the initial natural transformation

$$\Sigma^\infty_+\mathrm{Lace}^{\simeq}\longrightarrow F$$

whose target is a functor $F:\mathbf{Cat}^{\mathrm{lace}}$ such that each $F(\mathcal{C},-)$ is exact. Both of these approach are somewhat annoying to make precise. The reason is that the correct way to do this is the following:

Lemma 2.2.15 The functor $\widehat{\mathrm{tr}}$ coincides with the composite

$$\mathbf{Cat}^{\mathrm{lace}}\xrightarrow{\mathrm{can}}\operatorname*{lax\,colim}_{\mathcal{C}\in\mathbf{Cat}^{\mathrm{Ex}}}\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C},\mathrm{Sp})\longrightarrow\mathrm{Sp}$$

where the lax colimit is taken in $\mathbf{Cat}^{\mathrm{Ex}}$ (in particular, is a stable category) and the second map is the functor corepresented by the image of $(\mathrm{Sp}^{\mathrm{fin}},\mathrm{id})$ via the first functor. The first functor is the canonical comparison map from the oplax colimit computed in $\mathbf{Cat}$ and in $\mathbf{Cat}^{\mathrm{Ex}}$.

Moreover, $(\mathrm{Sp}^{\mathrm{fin}},\mathrm{id})$ is compact in the middle term hence, $\widehat{\mathrm{tr}}$ actually commutes with filtered colimits.

Proof. Essentially, the second recipe we gave to define $\widehat{\mathrm{tr}}$ is precisely how mapping spectra in the oplax colimit in $\mathbf{Cat}^{\mathrm{Ex}}$ are built. To make this proof more careful, we would have to invest a bit more time into oplax colimits in $\mathbf{Cat}^{\mathrm{Ex}}$, which we do not want to do for now; a reference is for instance [LNS25]. Let us also point towards Lemma **??** which proves a more general fact that (almost) recovers the one we want when specializing to $\mathcal{V}=\mathrm{Sp}$.

For the second part, recall that $\mathrm{Lace}^{\simeq}:\mathbf{Cat}^{\mathrm{lace}}\to\mathcal{S}$ preserves filtered colimits, as was shown during the process of proving Lemma 2.2.5 since this fact is equivalent to the compactness of $(\mathrm{Sp}^{\mathrm{fin}},\mathrm{id})$. Our claim is therefore essentially that the fiberwise stabilization process preserves filtered-colimit functors. This is clear from the formula: $\mathrm{colim}_n\,\Omega^nF(\mathcal{C},\Sigma^n)$. □

Proposition 2.2.16 There is an equivalence $\mathrm{cyc}(\widehat{\mathrm{tr}})(\mathcal{C},M)\simeq\mathrm{THH}(\mathcal{C},M)$ for every laced category $(\mathcal{C},M)$. In particular, THH is a well-defined functor on $\mathbf{Cat}^{\mathrm{lace}}$, which is trace-like and universally so out of $\widehat{\mathrm{tr}}$.

Proof. We first claim that the Bousfield–Kan formula applied to THH yields an equivalence:

$$\mathrm{THH}(\mathcal{C},M)\simeq\left|\dots\ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\ \operatorname*{colim}_{X\in T_1}M(Y_0,X_0)\ \substack{\longrightarrow\\\longrightarrow}\ \operatorname*{colim}_{X\in T_0}M(Y_0,X_0)\right|$$

where $T_n:=\mathrm{Fun}([n],\mathcal{C})^{\simeq}$ with the same notational convention as before Corollary 2.1.11. Again, we remark that the above formula is the edgewise subdivision of another, hence there is also an equivalence

$$\mathrm{THH}(\mathcal{C},M)\simeq\left|\dots\ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\ \operatorname*{colim}_{(X\to Y)\in\mathrm{Ar}(\mathcal{C})^{\simeq}}M(Y,X)\ \substack{\longrightarrow\\\longrightarrow}\ \operatorname*{colim}_{X\in\mathcal{C}^{\simeq}}M(X,X)\right|$$

Hence, to conclude, it suffices to prove that

$$\widehat{\mathrm{tr}}((\mathcal{C},M)^{([n],*)})\simeq\operatorname*{colim}_{X_0\to\dots\to X_n\in\mathrm{Fun}([n],\mathcal{C})^{\simeq}}M(X_n,X_0)$$

which is straightforward from the example following Proposition 2.2.9. □

Let us also record the following:

Corollary 2.2.17 The functor $\mathrm{THH} : \mathbf{Cat}^{\mathrm{lace}} \to \mathrm{Sp}$ preserves filtered colimits.

Proof. We know that $\widehat{\mathrm{tr}}$ commutes with filtered colimits by Lemma 2.2.15. Let us also note that each $(\mathcal{C}, M) \mapsto (\mathcal{C}, M)^{([n],*)}$ does by virtue of $([n], *)$ being compact in $\mathbf{Cat}^{\mathrm{unlace}}$: indeed this functor is adjoint to $(\mathcal{C}, M) \mapsto \mathrm{St}([n], *) \otimes (\mathcal{C}, M)$ where $\mathrm{St} : \mathbf{Cat}^{\mathrm{unlace}} \to \mathbf{Cat}^{\mathrm{lace}}$ is the functor we built in Proposition 2.2.9. The universal property shows that this functor preserves compacts as soon as $([n], *)$ is compact.

This latter claim is straightforward to check by hand: $[n]$ is compact in $\mathbf{Cat}$ (it is even finite!) and so is the constant functor $*$ equal to the terminal object since $*$ is compact in spaces, hence the explicit description of filtered colimits in $\mathbf{Cat}^{\mathrm{unlace}}$ (i.e. take the colimit underlying and then push the bimodule via left Kan extension) concludes.

It follows that the whole simplicial object $(\mathcal{C}, M)^{([\bullet],*)}$ commutes with filtered colimits and therefore its geometric realization, which is THH by the above Proposition. □

Remark 2.2.18 The upshot of the Proposition 2.2.16 is that the trace of a functor is not just the sum of all of the diagonal values, but one has to take into account the maps in the sum; in the same way that a natural transformation $F \implies G$ is not simply giving maps $F(X) \to G(X)$.

However, there is another to recover the real trace from the sum of diagonal values, it is to universally enforce a weak version of the cyclic invariance of the trace as in Remark 2.2.13.

Let us make a few remarks on what kind of trace invariance this implies for THH (and more generally trace-like invariant). By Proposition 2.2.12 and Remark 2.2.13, we see that for every adjunction $L : \mathcal{C} \dashv \mathcal{D} : R$ and $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$, we have a pair of laced functors

$$(\mathcal{C}, \mathrm{Ind}(R) \circ M) \rightleftarrows (\mathcal{D}, M \circ \mathrm{Ind}(R))$$

which are sent to inverses to one another by THH. Now, one can take M itself to be $\mathrm{Ind}(L)$ and under the further hypothesis that L has another left adjoint, apply this again but in the other direction with the bimodule being $\mathrm{Ind}(R)$ this time.

This gives two equivalences and therefore a $\mathbb{Z}$-action on say $(\mathcal{C}, \mathrm{Ind}(R) \circ \mathrm{Ind}(L))$ but since the two maps have a common inverse, this action must be trivial. It turns out that there is an interesting $\mathbb{Z}$-action on $\mathrm{THH}(\mathcal{C}, \mathrm{Ind}(R) \circ \mathrm{Ind}(L))$, in fact, it is a C_2-action, where C_2 is the cyclic group with two elements, but it is invisible to trace-like functors. Perhaps we will have time to explain this in a later section.

This leaving gaping a very sensible question: what happens if we try to enforce the real cyclic invariance property? Let us spoil already the surprise, THH is already cyclic invariant (hence the cyclic-invariant approximation of the naive trace). Now, proving this statement in its most structural version will occupy a lot of our later sections.

Let us first provide a version that is not quite fully coherent, but at least implement the idea. This can be done in many ways, classically through the cyclic Bar construction (but no reference in this language exist, as far as we are aware), and through the formalism of traces (see [KNP24]). We will present our own version, using coend magic, but all three suffer ultimately from the same defect: the equivalence is not induced by a map in $\mathbf{Cat}^{\mathrm{lace}}$. Our version is presumably the one for which this defect appears at the latest part.

Lemma 2.2.19 Let $M : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ and $N : \mathcal{D} \to \mathrm{Sp}$ be exact functors, $\mathcal{C}, \mathcal{D}$ both stable and let $f : \mathcal{C} \to \mathcal{D}$ be a functor . Then, f induces an equivalence in Sp:

$$\alpha(f) : \int^{\mathcal{C}} M \otimes f^*(N) \xrightarrow{\simeq} \int^{\mathcal{D}} f_!(M) \otimes N$$

where the tensor is pointwise the one of Sp and $f_!$ denotes left Kan extension along f, f^* is precomposition.

In consequence, the laced functor $(\mathcal{C}, M \otimes f^*(N)) \to (\mathcal{D}, f_!(M) \otimes N)$ is inverted by THH.

Proof. Let us first explain what is the induced map: there is natural transformation $\mathrm{id} \implies f_! f^*$

which induces a natural transformation of functors, and finally a map of coend

$$\int^{\mathcal{C}} M \otimes f^*N \longrightarrow \int^{\mathcal{C}} f_! f^*(M) \otimes f^*N$$

Moreover, there is also a map

$$\int^{\mathcal{C}} f_! f^*(M) \otimes f^*N \longrightarrow \int^{\mathcal{D}} f_!(M) \otimes N$$

induced by restricting the colimit along the induced $\mathrm{TwAr}(f)$. The composite is the wanted map.

Fix $Z \in \mathrm{Sp}$, we will show that the wanted map is an equivalence by showing that $\alpha(f)^*$ is an equivalence:

$$\mathrm{Map}(\int^{\mathcal{D}} f_!(M) \otimes N, Z) \longrightarrow \mathrm{Map}(\int^{\mathcal{C}} M \otimes f^*N, Z)$$

Pulling the limits out, We are equivalently trying to show that:

$$\int_{\mathcal{D}} \mathrm{Map}(f_!(M) \otimes N, Z) \longrightarrow \int_{\mathcal{C}} \mathrm{Map}(M \otimes f^*N, Z)$$

is an equivalence. Using that Sp is closed, this map identifies with the following

$$\int_{\mathcal{D}} \mathrm{Map}(f_!(M), \mathrm{map}(N, Z)) \longrightarrow \int_{\mathcal{C}} \mathrm{Map}(M, f^* \mathrm{map}(N, Z))$$

where $\mathrm{map}(N, Z)$ is notation for the functor $\mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ sending $X \in \mathcal{D}$ to $\mathrm{map}(N(X), Z)$ (so that the commutation with f^* on the right hand side is legitimate).

By Lemma 1.4.20, we have finally reduced this map to:

$$\mathrm{Nat}(f_!(M), \mathrm{map}(N, Z)) \longrightarrow \mathrm{Nat}(M, f^* \mathrm{map}(N, Z))$$

which is precisely the map that witness that $f_!$ and f^* are adjoint to one another by construction, so it is an equivalence as wanted. □

In different terms, what the above Lemma shows is that the laced functor

$$(\mathcal{C}, M \otimes f^*(N)) \longrightarrow (\mathcal{D}, f_!(M) \otimes N)$$

which is induced by $f : \mathcal{C} \to \mathcal{D}$ and the natural transformation $M \otimes f^*(N) \longrightarrow f^* f_!(M) \otimes f^*(N)$ itself induced by the counit of the adjunction between $f!$ and f^*, is inverted by THH. But note that the category $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{D}, \mathrm{Sp})$ is (compactly)-generated by the *split* bimodules, i.e. every exact functor $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathrm{Sp}$ is a filtered colimit of objects coming from the Yoneda embedding of $\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}}$ which are in particularly split since they of the form $\mathrm{map}_{\mathcal{C}}(-, X) \otimes \mathrm{map}_{\mathcal{D}}(Y, -)$ for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

Now note that the above map $(\mathcal{C}, M \otimes f^*(N)) \longrightarrow (\mathcal{D}, f_!(M) \otimes N)$ not is still defined for non-split bimodules but also is compatible with filtered colimits. In particular, since THH preserves filtered-colimits, see Lemma 2.2.17, we get:

Corollary 2.2.20 Suppose $f : \mathcal{C} \to \mathcal{D}$ is an exact functor, then for any colimit-preserving $M : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C})$, the laced functor

$$(\mathcal{C}, M \circ \mathrm{Ind}(f)) \longrightarrow (\mathcal{D}, \mathrm{Ind}(f) \circ M)$$

is inverted by THH.

In the case where f has a right adjoint, we are also in the situation of Proposition 2.2.12. Proposition 2.2.16 implies both maps possibly appearing in in Corollary 2.2.20 (where f is either the left or right adjoint) are sent to inverse to one another by THH.

Remark 2.2.21 Suppose in the situation of the Corollary that $M = \mathrm{Ind}(g)$ for some $g : \mathcal{D} \to \mathcal{C}$. Then, we have found two equivalences and therefore a $\mathbb{Z}$-action on $\mathrm{THH}(\mathcal{C}, \mathrm{Ind}(f) \circ \mathrm{Ind}(g))$ and this time, there is no common inverse in sight to come and be meddlesome. But alas, it so happens that the two equivalences are still inverses to one another; this can be gruesomely deduced from the above Lemma but is also very conceptually clear in a generalization we will present later.

Note however that not everything is lost: if $f = g$, then the fact that the composite

$$\mathrm{THH}(\mathcal{C}, \mathrm{Ind}(f) \circ \mathrm{Ind}(g)) \xrightarrow{\eta_f} \mathrm{THH}(\mathcal{C}, \mathrm{Ind}(g) \circ \mathrm{Ind}(f)) \xrightarrow{\eta_g} \mathrm{THH}(\mathcal{C}, \mathrm{Ind}(f) \circ \mathrm{Ind}(g))$$

is actually witnessing a C_2-action on $\mathrm{THH}(\mathcal{C}, \mathrm{Ind}(f)^{\circ 2})$! This is nothing specific about $n = 2$ and being brave, one could possibly undertake to check by this principle that there is an action of the cyclic group with n elements C_n on $\mathrm{THH}(\mathcal{C}, \mathrm{Ind}(f)^{\circ n})$.

We can do better than the above: the category $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$ is compactly-generated by the image of $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})$ under Ind (this is in fact *verbatim* the previous claim up to the usual bimodules to continuous Ind-functor equivalence) so we can also replace $\mathrm{Ind}(f)$ by a generic bimodule:

Proposition 2.2.22 The functor $\mathrm{THH} : \mathbf{Cat}^{\mathrm{lace}} \to \mathrm{Sp}$ is *trace-invariant*, i.e. comes with a canonical equivalence

$$\mathrm{THH}(\mathcal{C}, M \otimes_{\mathcal{D}} N) \simeq \mathrm{THH}(\mathcal{D}, N \otimes_{\mathcal{C}} M)$$

for $\mathcal{C}, \mathcal{D}$ stable and $M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathrm{Sp}$ and $N : \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$. Moreover, this equivalence comes from a map in $\mathbf{Cat}^{\mathrm{lace}}$ when the bimodules are split.

Warning 2.2.23 We have lost something very important in this extension however: there is no longer a map in $\mathbf{Cat}^{\mathrm{lace}}$ that is getting inverted, since maps in $\mathbf{Cat}^{\mathrm{lace}}$ have to come from a map $\mathcal{C} \to \mathcal{D}$. This makes the definition of *trace invariants* feel a bit icky for the homotopy theorist, unless one has an extra assumption of preserving filtered colimits, in which case one can simply ask the map of Corollary 2.2.20 to be inverted

Unfortunately, we will also be interested in invariants which need not commute with filtered colimits — and even if one could also cheat in this case because they factor through some refinement of THH, this is getting too ugly for our pure souls. In a later section, we will find a way to get this equivalence induced from an actual map, and doing this properly will additionally shed some light onto what is going on with the action of Remark 2.2.21.

There is even more one can say. Recall that for matrices over a commutative ring R, the trace is the unique, up to a multiplicative constant, function $M_n(R) \to R$ which is both invariant under cyclic permutations and linear. In his thesis [Ram24], Ramzi proved this characterization is robust enough to lift to higher categories:

Theorem 2.2.24 — Ramzi. Let $\mathcal{E}$ be presentable stable and $\mathrm{Tr}^{\mathrm{L},\omega}(\mathbf{Cat}^{\mathrm{lace}}, \mathcal{E})$ denote the full category of functors $F : \mathbf{Cat}^{\mathrm{lace}} \to \mathcal{E}$ such that

(i) F preserves filtered colimits

(ii) For every $\mathcal{C} \in \mathbf{Cat}^{\mathrm{Ex}}$, the restriction $F(\mathcal{C}, -)$ preserves filtered colimits

(iii) F inverts the map of Corollary 2.2.20

Then, evaluation at $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$ induces an equivalence

$$\mathrm{ev}_{(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})} : \mathrm{Tr}^{\mathrm{L},\omega}(\mathbf{Cat}^{\mathrm{lace}}, \mathcal{E}) \xrightarrow{\simeq} \mathcal{E}$$

whose inverse is given by $X \in \mathcal{E} \longmapsto X \otimes \mathrm{THH}(-)$, where $\otimes$ denotes the tensoring of the presentable stable $\mathcal{E}$ by Sp.

Proof. Recall that the fibre of $\mathbf{Cat}^{\mathrm{lace}}$ over a general $\mathcal{C}$, namely $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ is generated under filtered colimits by those functors $\operatorname{Ind}\mathcal{C} \to \operatorname{Ind}\mathcal{C}$ which factor as

$$\operatorname{Ind}(\mathcal{C}) \xrightarrow{\operatorname{map}(-,X)} \mathrm{Sp} \xrightarrow{Y\otimes -} \operatorname{Ind}(\mathcal{C})$$

for $X, Y \in \mathcal{C}$, where the Yoneda embedding has been suppressed from the notation. This is a consequence of the identification $\operatorname{Ind}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \simeq \operatorname{Fun}^{\mathrm{L}}(\operatorname{Ind}(\mathcal{C}), \operatorname{Ind}(\mathcal{C}))$ which we have already used under multiple different names.

But note that for such bimodules, by (iii), F gives an equivalence

$$F(\mathcal{C}, Y \otimes \operatorname{map}_{\mathcal{C}}(-,X)) \simeq F(\mathrm{Sp}^{\mathrm{fin}}, \operatorname{map}_{\mathcal{C}}(Y,X) \otimes -)$$

where f is $Y \otimes -$ (this uses the compactness assumption on Y) and M corresponds to $\operatorname{map}(-,X)$ Since $\operatorname{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Sp}$ is generated under colimits by the identity, the linearity hypothesis (ii) on each F implies that the colimit-comparison map

$$\operatorname{map}_{\mathcal{C}}(Y,X) \otimes F(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}) \longrightarrow F(\mathrm{Sp}^{\mathrm{fin}}, \operatorname{map}_{\mathcal{C}}(Y,X) \otimes -)$$

is an equivalence. This concludes by the first point. □

Let us first make a remark about whether the above theorem has the minimal assumptions (it does not):

Remark 2.2.25 Assertion (i) is not quite used in its full potential, since we actually only need that $F(\mathcal{C}, -)$ preserved filtered colimits. But given the expression of the inverse, this is a proof that this assertion is implied by the other; this can also be seen more directly.

In a similar vein, we did not use that $F(\mathcal{C}, -)$ was exact except when $\mathcal{C} = \mathrm{Sp}^{\mathrm{fin}}$, but it follows and can be deduced directly from straightforward manipulations.

A second remark we want to make is that actually the proof has split the statement in two halves of independent interest:

Remark 2.2.26 In fact, the above proof has asserted something ever-so-slightly more general: every trace-invariant functor which is filtered-colimit preserving in the bimodule variable is fully-determined by its restriction $F(\mathrm{Sp}^{\mathrm{fin}}, -) : \mathrm{Sp} \to \mathrm{Sp}$, which is also finitary hence determined by $F(\mathrm{Sp}^{\mathrm{fin}}, -) : \mathrm{Sp}^{\mathrm{fin}} \to \mathrm{Sp}$.

When $F(\mathrm{Sp}^{\mathrm{fin}}, -)$ is exact, i.e. of the form, $X \otimes -$ for $X \in \mathrm{Sp}$, we have an inverse to this process, which is namely to extend the functor to the whole of $\mathbf{Cat}^{\mathrm{lace}}$ as $X \otimes \mathrm{THH}$. Note that there is no longer an inverse if we lose the exactness, at least without extra assumption, as $\mathrm{THH}(\mathcal{C}, M^{\otimes 2})$ and $\mathrm{THH}(\mathcal{C}, M)^{\otimes 2}$ show. We will come back to those issues later.

One Corollary of Theorem 2.2.24 is as follows:

Corollary 2.2.27 The functor $\mathrm{THH} : \mathbf{Cat}^{\mathrm{lace}} \to \mathrm{Sp}$ is the initial such functor which is lax-monoidal, trace-invariant, colimit-preserving in the bimodule variable. Moreover, this lax-monoidal structure is actually strong monoidal.

Proof. Note that both $\mathrm{THH}((\mathcal{C}, M) \otimes -)$ and $\mathrm{THH}(\mathcal{C}, M) \otimes \mathrm{THH}(-)$ are trace-invariant, colimit-preserving functors in the bimodule variable. The only non-trivial claim is that $\mathrm{THH}((\mathcal{C}, M) \otimes -)$ is indeed trace-invariant, but this is because our definition of trace-invariance is quite ad-hoc.

Given N, P respectively a $(\mathcal{D}, \mathcal{E})$ and $(\mathcal{E}, \mathcal{D})$-bimodule, then $M \boxtimes N$ and $M \boxtimes P$ are $(\mathcal{C} \otimes \mathcal{D}, \mathcal{C} \otimes \mathcal{E})$ and $(\mathcal{C} \otimes \mathcal{E}, \mathcal{C} \otimes \mathcal{D})$-bimodule. In particular, the trace-invariant of THH applied to those bimodules gives precisely the wanted claim.

Therefore, Theorem 2.2.24 provides a canonical equivalence

$$\mathrm{THH}(\mathcal{C}, M) \otimes \mathrm{THH}(\mathcal{D}, N) \simeq \mathrm{THH}((\mathcal{C}, M) \otimes (\mathcal{D}, N))$$

which is natural in both variables, which upgrades to a symmetric monoidal structure on THH. In fact, we get far more from the above argument: any map

$$F(\mathcal{C}, M) \otimes F(\mathcal{D}, N) \simeq F((\mathcal{C}, M) \otimes (\mathcal{D}, N))$$

for $F : \mathbf{Cat}^{\mathrm{lace}} \to \mathrm{Sp}$ a trace-invariant, fiberwise-colimit preserving functor $F : \mathbf{Cat}^{\mathrm{lace}} \to \mathrm{Sp}$ corresponds essentially uniquely to a map $F(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}) \otimes F(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}) \to F(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$ in Sp. More generally, this upgrades to show that a lax-symmetric monoidal structure corresponds to a $\mathbb{E}_\infty$-structure on $F(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$. But $\mathbb{S}$ is the initial such object, hence the first claim of the Corollary. □

We will eventually show more, i.e. that THH is still initial lax-monoidal for a weaker property (namely the one hinted at by Proposition 2.2.16).

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